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# NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

## TECHNICAL MEMORANDUM 1234

## COMPUTATION OF THIN-WALLED PRISMATIC SHELLS

By V. Z. Vlasov

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# COMPUTATION OF THIN-WALLED PRISMATIC SHELLS*

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## 1. Fundamental Assumptions

We consider a prismatic shell consisting of a finite number of narrow rectangular plates and having in the cross-section a finite number of closed contours (fig. 1(a)). We shall assume that the rectangular plates composing the shell are rigidly joined so that there is no motion of any kind of one plate relative to the others meeting at a given connecting line. The position of a point on the middle prismatic surface is considered to be defined by the coordinate $z$, the distance to a certain initial cross-section $z = 0$, and the coordinate $s$ determining its position on the contour of the cross-section.

Let the function $u(z,s)$ represent a longitudinal displacement of the point $(z,s)$, that is, a displacement in the direction of the generator (positive in the direction of increasing $z$) and the function $v(z,s)$ represent a tangential displacement, that is, a displacement in the direction of the tangent to the contour of the cross-section (positive in the direction of increasing $s$).

We shall represent these displacements in the form of the sums:

$$u(z,s) = \sum_{i}^{m} U_i(z)\varphi_i(s) \qquad (i = 1,2,\ldots, m) \tag{1.1}$$

$$v(z,s) = \sum_{i}^{n} V_k(z)\psi_k(s) \qquad (k = 1,2,\ldots, n) \tag{1.2}$$

where the functions $U_i(z)$ and $V_k(z)$ depending only on $z$ are the required functions and the functions $\varphi_i(s)$ and $\psi_k(s)$ are subject to a preliminary choice. The magnitudes $m$ and $n$ will be explained below.

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*"Raschet Tonkostennikh Prismaticheskikh Obolochek." Prikladnaya Matamatika i Mekhanika, Vol. 8, No. 5, 1944.

We consider an elementary tranverse lamina cut out by the planes $z = \text{constant}$ and $z + dz = \text{constant}$. Such a lamina will be regarded as a plane system of rods, that is, a frame consisting of closed contours.

Let us consider the deformed state of this lamina determined only by the longitudinal displacements $u(z,s)$. The plane contour of such an elementary frame, while remaining on the prismatic surface of the shell, goes over into a space line that can be determined relative to the initial section $z = \text{constant}$ for the chosen functions $\varphi_i(s)$ by equation (1.1). We make the assumption that, in the case of the deformed state determined only by the longitudinal displacements, the rectilinear elements of the frame, starting from the plane $z = \text{constant}$, remain straight. Such an assumption is equivalent to the hypothesis of plane sections assumed individually for each of the narrow rectangular plates composing the given shell. In this case the position of an elementary frame after deformation is entirely determined by the longitudinal displacements of its $m$ joints relative to the plane $z = \text{constant}$. Hence the elementary frame can be considered as a rod system possessing relative to the longitudinal displacements $m$ degrees of freedom.

We shall take in equation (1.1) the required functions $U_1(z)$, $U_2(z)$, ..., $U_m(z)$ as the longitudinal displacements of the $m$ joints of an elementary frame. The functions $\varphi_1(s)$, $\varphi_2(s)$, ..., $\varphi_m(s)$ corresponding according to equation (1.1) to these displacements satisfy all the required conditions of the continuity of the longitudinal displacements $u(z,s)$ over the section $z = \text{constant}$.

Each of the functions $\varphi_i(s)$ for a choice of the required magnitudes $U_i(z)$ thus has, by the method here indicated, a very simple geometrically evident expression; namely, it is different from zero only on rectilinear parts of the contour meeting at the joint $i$; within the limits of each of these segments it is represented as a function of $s$ by a linear function of $s$ and assumes a value equal to unity at a point coinciding with the ith joint, becoming zero at the other end point of the given segment. On all the remaining parts of the contour line the function $\varphi_i(s)$ will be identically equal to zero (fig. 2).

It is clear that the chosen method of constructing the functions $\varphi_i(s)$ for the assumed model is not the only one. For the required functions $U_i(z)$ there may be taken any $m$ independent magnitudes. To each aggregate of $m$ independent magnitudes $U_i(z)$ will then correspond an aggregate of $m$ linearly independent functions $\varphi_i(s)$ each of which will be continuous over the entire multiply connected contour and on an individual segment of the contour is represented by a linear graph. Thus, for example, if we wish in the first of the series (1.1) to separate initially the longitudinal displacements relative to the elementary computation of the shell as a beam of a composite multiply connected cross-section on

the basis of the hypothesis of Bernoulli assumed for the entire section $z = \text{constant}$, we must, for three of the $m$ possible displacements $U_i(z)$, take the values $U_1(z)$, $U_2(z)$, and $U_3(z)$ determining the displacements of the hinged model on the prismatic surface as a rigid plane system. The functions $\varphi_1(s)$, $\varphi_2(s)$, and $\varphi_3(s)$ corresponding to these magnitudes are each linear functions of the cartesion coordinates $x = x(s)$ and $y = y(s)$ of a point of the contour of the cross-section of the shell. The remaining terms of the series (1.1) for the function $u(z,s)$ determine a state of longitudinal displacements for which the cross-sections of the shell, after deformation, do not remain plane. Retaining the terminology adopted in our work on the general theory of thin-walled rods (reference 1) we shall denote the deviation from the law of plane cross-sections as "deplanations" of cross-section.

The deplanation of a multiply connected shell is thus determined by the $m - 3$ independent quantities $U_4(z), U_5(z), \ldots, U_m(z)$. These magnitudes will be called the generalized coordinates of the deplanation.

In constructing the functions $\psi_k(s)$ $(k = 1,2,3, \ldots, n)$ entering equation (1.2) for the transverse tangential displacement $v(z,s)$ of a point of the contour $(z,s)$, we shall start from the deformation of an elementary transverse lamina (fig. 1(a)) in its plane $z = \text{constant}$. Considering this lamina as a rod system and assuming the elements of the frame to be inextensible we arrive at the result that the contour displacement $v(z,s)$ may be expressed in terms of the displacements $V_k(z)$ $(k = 1,2,3, \ldots, n)$ of the plane kinematic model considered above in the plane of the cross-section of the shell. The coefficients of $V_k(z)$ of the sum (1.2) can be taken as independent magnitudes determining the form of the displacements of the hinged rod system in its plane. The number $n$ of required functions $V_k(z)$ is equal to the number of degrees of freedom of this system in the plane of the cross-section and is determined by the formula

$$n = 2m - c \tag{1.3}$$

where $m$ is the number of joints and $c$ is the number of rods of the transverse multiply connected elementary frame.

By choosing in some manner $n$ independent magnitudes $V_k(z)$ for the displacements of an elementrary rod system in the plane of the section $z = \text{constant}$ and giving in succession to each of these magnitudes unit values and assuming the others equal to zero we can, by considering the elementary displacements thus obtained of this system, determine all the required functions $\psi_k(s)$. Each of these functions will represent a contour displacement of a point $s$ in correspondence with the elementary state $V_k{}^* = 1$ and $V_h{}^* = 0$ for $h \neq k$ (fig. 3). The function $\psi_k(s)$ within the limits of each straight segment of the contour of the shell maintains a constant value (does not depend on $s$) and represents an axial displacement of the corresponding hinged rod model. We shall

thus set up $n$ linearly independent diagrams of the functions $\psi_k(s)$ for any choice of the magnitudes $V_z(z)$.

These magnitudes will be chosen in such a manner that three of them, $V_1(z)$, $V_2(z)$, and $V_3(z)$, will refer to the displacements of the model as a plane system of rods as a whole without change in shape of this system. The remaining magnitudes $V_4(z)$, $V_5(z)$, ..., $V_n(z)$ will refer to such displacements of the system for which the mutual position of the individual links of the system changes. The phenomena associated with the change in shape of the hinged rod system will be denoted as the deformation of the contour of the system. The contour deformation is thus determined by $n - 3$ independent magnitudes where $n$ is the number of degrees of freedom of an elementary lamina of the shell considered as a plane hinge-connected system. The magnitudes $V_4(z)$, $V_5(z)$, ... $V_n(z)$ will be denoted as the generalized coordinates of the deformation of the shell contour.

The functions $\psi_k(s)$ chosen in some way or other corresponding to the $n$ degrees of freedom of the system of rods in its plane satisfy the condition of linear independence and the condition of continuity of the displacements determined by these functions of the elementary transverse frame at all points of its contour including also the nodal points since the hinged model in each of the $n$ possible elementary states $V_k^* = 1$, $V_h^*(z) = 0$ for $h \neq k$ remains everywhere continuous.

## 2. Fundamental Differential Equations – Method of Displacements

After choosing the functions $\varphi_i(s)$ and $\psi_k(s)$ in the sums (1.1) and (1.2) the problem reduces to the determination of the functions $U_i(z)$ and $V_k(z)$ $(i = 1,2,3, \ldots, m, \quad k = 1,2,3, \ldots, n)$.

Let $\sigma = \sigma(z,s)$ and $\tau = \tau(z,s)$ denote respectively the normal and tangential stresses arising in the section $z$ = constant. We shall consider these stresses functions only of the coordinates of the point $(z,s)$ on the surface assuming that over the thickness of the shell the stresses $\sigma$ and $\tau$ are uniformly distributed. On the basis of Hooke's law

$$\sigma = E\frac{\partial u}{\partial z}, \qquad \tau = G\left(\frac{\partial u}{\partial s} + \frac{\partial v}{\partial z}\right) \tag{2.1}$$

Hence making use of equations (1.1) and (1.2) we have

$$\sigma(z,s) = E\sum U_i'(z)\varphi_i(s) \qquad (i = 1,2,3, \ldots, m) \tag{2.2}$$

$$\tau(z,s) = G\left[\sum U_i(z)\varphi_i'(s) + \sum V_k'(z)\psi_k(s)\right] \quad (i = 1,2,\ldots, m, k = 1,2,\ldots, n) \tag{2.3}$$

where E and G are the moduli of elasticity of the material of the shell in tension and shear.

The elementary cross-sectional lamina of the shell will be under the action of external forces consisting of normal and shearing forces acting in the sections $z = \text{constant}$ and $z + dz = \text{constant}$ and the given surface forces. Let $p^* = p^*(z,s)$, and $q^* = q^*(z,s)$ denote the forces external with respect to the given lamina acting respectively along the generator of the shell (positive in the direction of increasing coordinate $z$) and tangential to the contour line of the cross-section (positive in the direction of increasing coordinate $s$). Referring these forces to unit area of the middle surface we obtain the equations

$$p^* = \frac{\partial\sigma}{\partial z}\delta + p \qquad q^* = \frac{\partial\tau}{\partial z}\delta + q \tag{2.4,2.5}$$

where the thickness of the shell $\delta = \delta(s)$ is assumed a given (in the general case not continuous) function of a single coordinate $s$ and the magnitudes $p = p(z,s)$, and $q = q(z,s)$ represent the given external surface forces.

The integral conditions of equilibrium of the elementary frame for chosen forms of the displacement determined by the $m + n$ degrees of freedom, on the basis of the principle of Lagrange can be represented in the form of $m + n$ equations:

$$\int \frac{\partial\sigma}{\partial z}\varphi_j dF - \int \tau\varphi_j' dF + \int p\varphi_j ds = 0 \qquad (j = 1,2,3,\ldots, m) \tag{2.6}$$

$$\int \frac{\partial\tau}{\partial z}\psi_h dF - \sum V_k \int \frac{M_k M_h}{EJ} ds + \int q\psi_h ds = 0 \qquad (h = 1,2,3,\ldots, n) \tag{2.7}$$

where $dF$ is the differential of the area of the shell cross-section

$$dF = \delta ds \tag{2.8}$$

The integrals above and those given in the following are definite and are taken over the entire contour of the section $z = \text{constant}$.

Equations (2.6) represent $m$ conditions of equilibruim of the lamina $dz = 1$ in the direction perpendicular to the plane $z = \text{constant}$; equations (2.7) represent $n$ conditions of equilibrium of the same lamina in the plane $z = \text{constant}$. Each of the equations (2.6) expresses the equating to zero of the sum of the work of all external and internal forces of an elementary lamina with change in the deformed state of the lamina relative to its plane.

For the virtual displacements in the equations of subscript $j$ of this group there are taken the longitudinal displacements $u_j = \varphi_j(s)$ of the points of the elementary frame determined only by the term of subscript $j$ of the sum (1.1) for $U_j{}^* = 1$. The first and third terms of this equation refer to the work of the external forces $p^* = \frac{\partial\sigma}{\partial z}\delta + p$ acting on the element of width $dz = 1$ and directed perpendicular to its plane.

The middle term expresses the work of the internal shearing forces. For an element $ds$ this work is determined as the product (with reversed sign) of the shearing force $\tau\delta$ **ds** by the shear deformation which in the case considered of the variation of the deformed state is equal to the derivative $\varphi_j{}'(s)$ of the function $\varphi_i(s)$.

Each of equations (2.7) was obtained by equating to zero the sum of the work of all the external and internal forces of the elementary frame on the corresponding displacements for a change in the deformed state of the element in its plane. For the virtual displacements in the equation of subscript $h$ there were taken the transverse contour displacements $v_h = \psi_h(s)$ of the element determined only by the term of subscript $h$ of the sum (1.2) for the generalized coordinate $V_h{}^* = 1$. The first and third terms of equations (2.7) refer to the work of the external contour forces $q^* = \frac{\partial\tau}{\partial z}\delta + q$ of the element acting in its plane. The second term expresses the work of the internal forces on the deformations of the element corresponding to the h-th elementary state of the displacements of the hinged system in its plane. For the element $ds$ this work in the case of bending is determined (with reversed sign) as the product of the bending moment

$$M(z,s) = \sum V_k(z)M_k(s) \qquad (k = 1,2,3, \ldots, n) \tag{2.9}$$

by the angle of rotation $\frac{M_h(s)}{EJ}ds$ of two adjacent sections bounding this element. The letters $M_k = M_k(s)$ and $M_h = M_h(s)$ denote the bending

moments of the transverse frame element corresponding to the elementary states of deformation of this frame $V_k^* = 1$ and $V_h^* = 1$. These moments are found by the usual methods of structural mechanics by proceeding backward from the deformed state of the rod system to the internal forces.

The magnitude $J = J(s)$ represents the moment of inertia of an arbitrary cross-section of the elementary frame of the shell having a width $dz = 1$. If the shell consists of only rectangular plates and has no transverse connections then evidently

$$J = \frac{\delta^3}{12} \tag{2.10}$$

where $\delta$ is the thickness of the corresponding plate.

In the case of a shell reenforced with additional transverse frames the magnitude $J$ must be computed with account taken of the mean moment of inertia of these frames, that is the moment of inertia associated with a unit length of the shell.

Substituting in equations (2.6) and (2.7) for $\sigma$ and $\tau$ their expressions (2.2) and (2.3) we obtain a system of $m + n$ linear differential equations with respect to the required generalized displacements, $m$ longitudinal $U_i(z)$ $(i = 1,2,\ldots, m)$ and $n$ transverse $V_k(z)$ $(k = 1,2,\ldots, n)$. This system can be represented in the form

$$\left.\begin{aligned} &\gamma \sum_i a_{ji}U_i'' - \sum_i b_{ji}U_i - \sum_k c_{jk}V_k' + \frac{1}{G} p_j = 0 \\ &\sum_i c_{hi}U_i' + \sum_k r_{hk}V_k'' - \gamma \sum_k s_{hk}V_k + \frac{1}{G} q_h = 0 \end{aligned}\right\} \quad \begin{pmatrix} i,j = 1,2,\ldots, m \\ h,k = 1,2,\ldots, n \end{pmatrix} \tag{2.11}$$

where $\gamma$ is a constant magnitude defined by the equation

$$\gamma = \frac{E}{G} \tag{2.12}$$

The coefficients of equations (2.11) are computed by the formulas:

$$a_{ji} = \int \varphi_j(s)\varphi_i \, dF, \quad c_{jk} = \int \varphi_j'(s)\psi_k(s) \, dF, \quad r_{hk} = \int \psi_h(s)\psi_k(s) \, dF$$

$$b_{ji} = \int \varphi_j'(s)\varphi_i'(s) \, dF, \quad c_{hi} = \int \psi_h(s)\varphi_i'(s) \, dF, \quad s_{hk} = \int \frac{M_h(s)M_k(s)}{EJ} \, ds \tag{2.13}$$

where the integrals are taken over all the elements of the cross-section of the shell. These coefficients possess the properties of symmetry

$$\begin{matrix} a_{ji} = a_{ij}, & r_{hk} = r_{kh} \\ b_{ji} = b_{ij}, & s_{hk} = s_{kh} \end{matrix} \quad c_{hi} = c_{jk} \quad \text{for} \quad h = k \tag{2.14}$$

expressed by the theorem of Betti on the reciprocity of the work of an elastic system.

Formulas (2.13) are of a general character and permit computing the coefficients of the equations (2.11) for a shell of arbitrary contour of cross-section for any method of approximation of the required displacements $u(z,s)$, $v(z,s)$ for varying $s$.

In choosing the functions $\varphi_i(s)$ $(i = 1,2,\ldots, m)$, $\psi_k(s)$ $(k = 1,2,\ldots, n)$ by the method described above the quadratures on the right sides of formulas (2.13) for each straight segment of the contour receive a simple expression because each of the functions $\varphi_i$ over this segment depends on the coordinate $s$ linearly and the derivative $\varphi_i'$, like each of the functions $\psi_k$, has a constant value over a straight part of the contour.

The quadratures of the first five formulas (2.13) for $dF = \delta \, ds$ have the same form as the last of these formulas except that instead of the magnitudes reciprocal to the moment of inertia the thickness of the shell enters under the integral sign. All the coefficients of the equations can be computed by the known devices of the theory of framed structures with the aid of diagrams of the functions $\varphi_i(s)$, $\varphi_i'(s)$, $\psi_k(s)$, $M_k(s)$ constructed for the entire multiply connected contour.

The magnitudes $p_j(s)$ and $q_h(z)$ referring to the free terms of equations (2.11) are known functions of $z$ and for given surface forces of the shell $p(z,s)$, $q(z,s)$ are computed by the formulas

$$p_j = \int p\varphi_j \, ds, \qquad q_h = \int q\psi_h \, ds \tag{2.15}$$

where the contour integrals are taken over the parts of the contour for which the expressions under the integral sign are different from zero.

The magnitudes $p_j(z)$ and $q_h(z)$ corresponding to their physical meaning arising from the method of their definition can be termed the unit (referring to unit length of the shell) generalized external forces. The longitudinal one, $p_j(z)$, is computed as the work of the external longitudinal surface forces, $p(z,s)$, over the longitudinal displacements, $\varphi_j(s)$, of the elementary (unit) state of deformation of the lamina for $U_j^* = 1$, while the transverse one, $q_h(z)$, is computed as the work of the external surface contour forces $q(z,s)$ obtained for the contour displacements of the points of the elementary lamina $\psi_h(s)$, determined by the unit displacement $U_h^* = 1$.

Formulas (2.13) and (2.15) are easily extended also to shells stressed over individual longitudinal elements (stringers) and under the action of concentrated loads in the section $z$ = constant. In this case the quadratures in (2.13) and (2.15) must be taken in the sense of the Stieltjes integrals. To the integrals for continuous distribution over the contour of the differentials $dF$, $p\,ds$, $q\,ds$ must be added magnitudes which represent the sum of the products of the finite factors concentrated at definite points (the areas of the stringers, the concentrated forces) and the values of the corresponding functions under the integrals at these points.

The systems of the fundamental differential equations (2.11) of the prismatic shell may be represented in the form of table 1 (2.16) (found at the end of the text). This table represents the differential matrix and the free terms of equations (2.11). The elements of the matrix represent the linear differential operators with constant coefficients, where these operators in the first and fourth quadrants are of the second order and in the second and third quadrants are of the first order ($D$ and $D^2$ denote the derivatives with respect to $z$ of the first and second order of the corresponding required function written in the top row over the column containing the given differential element of the matrix). To obtain any differential equation of the system it is necessary to multiply each of the elements of the corresponding row of the matrix symbolically (by which is meant the differential operation expressed by the given element) by the function in the top row over the column containing the given element, combine all these products and to the sum add the free term of the given row and equate the result to zero. The system of differential equations represented in the form of table 1 consists of $m + n$ ordinary linear differential equations with constant coefficients each of the second order with respect to the $m + n$ required functions $U_i(z)$ and $V_k(z)$.

This system of equations by its physical meaning divides itself into two groups of equations. The first group written out in the first half of the table consists of m equations and expresses the equilibrium of the elementary elastic frame lamina out of its plane as a system of rods possessing on the prismatic surface m degrees of freedom. The second group of equations, consisting of n equations, refers to the equilibrium of this frame in its plane possessing in the latter n degrees of freedom. The matrix of the differential equations possesses a symmetrical structure and consists of four quadrants of which the first and fourth are the principal quadrants, the second and third the secondary, and represent energostatically the mutual effect of the two forms of deformations of the elastic shell. The elements of the secondary quadrants of the matrix which are symmetrical with respect to the principal (diagonal) terms differ only in sign. The equality of these elements in absolute value is a consequence of the theorem on the reciprocal work of elastic systems.

The equations here described represent a generalization of the earlier obtained eight-term equations for cylindrical shells and composite systems of open profile (references 1 and 2). The equations represented in the form of table 1 are completely analogous to the equations of the theory of statically indeterminate systems and differ from the latter in that these equations are differential since the deformed state of the transverse plane frame as an element of the shell depends on the position of the element along the length of the shell (on the coordinate $z$).

The differential equations (2.11) were derived for an arbitrary choice of the functions $\varphi_i(s)$ and $\psi_k(s)$ determining according to equation (2.13) the coefficients of these equations. Since the functions $\varphi_i$ and $\psi_k$ are linearly independent and each of them may be given with an accuracy up to an arbitrary factor, for the required functions $U_i(z)$ and $V_k(z)$ there can always be chosen such independent generalized longitudinal and transverse displacements of the elementary hinged system of the shell $dz = 1$ for which the polygonal functions $\varphi_i(s)$ and $\psi_k(s)$ corresponding to these displacements over the entire cross-section $z$ = constant possess the property of orthogonality. Assuming each of the pair of functions $\varphi_i(s)$ and $\psi_k(s)$ orthogonal, we shall have

$$a_{ji} = \int \varphi_j\varphi_i \, dF = 0, \quad \text{if} \quad j \neq i$$

$$r_{hk} = \int \psi_h\psi_k \, dF = 0, \quad \text{if} \quad h \neq k \tag{2.17}$$

For these conditions equations (2.11) assume the form

$$\gamma a_{jj}U_j'' - \sum_i b_{ji}U_i - \sum_k c_{jk}V_k' + \frac{1}{G} p_j = 0 \qquad (i,j = 1,2,\ldots, m)$$

$$\sum_i c_{hi}U_i' + r_{hh}V_h'' - \gamma \sum_k s_{hk}V_k + \frac{1}{G} q_h = 0 \qquad (h,k = 1,2,\ldots, n) \qquad (2.18)$$

The differential matrix of these equations now has a simpler structure since, with the condition of orthogonality (2.17) satisfied, the secondary differential terms of the principal (first and fourth) quadrants, table 1, that is the terms with the coefficients $a_{ij}$ and $r_{hk}$, for $i \neq j$, $h \neq k$ vanish.

Written out in full equations (2.18) are represented in table 2 (2.19) (found at the end of the text). The magnitudes $U_i(z)$ and $V_k(z)$ corresponding to the conditions of orthogonality may be called the principal generalized coordinates of the longitudinal and transverse displacements of the shell.

The choice of the orthogonal functions $\varphi_i(s)$ and $\psi_k(s)$ may be made by the graphico-analytical methods of structural mechanics by drawing the diagrams of these functions and orthogonalizing the group states of the elementrary displacements of the transverse laminar element.

## 3. Boundary Effect – The Internal Generalized Forces – Longitudinal and Transverse Bimoments

The equations (2.19) or, in the case of an arbitrary choice of the functions $\varphi_i$ and $\psi_k$, equation (2.16) possess the same structure as the equations of vibration of an elastic system possessing a finite number of degrees of freedom. Of the present day methods of integration of a symmetrical system of linear differential equations with constant coefficients the most effective method is that of A. N. Krylov, which permits reducing this system rapidly to its equivalent single differential equation. In our case this equation will be of the order $2(m + n)$. It follows that the required functions $U_i(z)$ and $V_k(z)$ satisfying the system of equations (2.18) will be determined with an accuracy up to $2(m + n)$ arbitrary constants. The number of these constants is equal to twice the number of degrees of freedom of the elementary lamina of the shell $dz = 1$ in space. This is in full agreement with the number of

independent kinematic conditions which can be given for the end sections of the shell $z = 0$, $z = l$ (where $l$ is the length of the shell in the direction of the generator). The position of all the points of any of these sections in space for a shell treated in the section $z$ = constant as a discrete system is determined by $m + n$ independent magnitudes of which $m$ magnitudes $U_i(z)$ determine the position of all points of juncture of the section on a prismatic surface (along the generator of the shell) and $n$ magnitudes $V_k(z)$ determine the position of these points in the plane of the cross-section. For a single end section of the shell $m + n$ arbitrary magnitudes may thus be assigned. For the two sections $z = 0$ and $z = l$ which bound the given shell over its length the number of independent conditions is equal to $2(m + n)$ which corresponds to the number of arbitrary constants in the integrated equations (2.11).

With these constants given we can obtain a solution for a given shell for the most varied boundary conditions relative to the longitudinal and transversed displacements, this solution being entirely determinate and unique and satisfying all the required kinematic conditions of the shell.

Let us now consider the problem of the equilibrium of the shell for which the boundary conditions in the sections $z = 0$ and $z = l$ are given in terms of stresses or, in the case of the mixed boundary problem, partly in terms of stresses and partly in terms of displacements. By definition of the functions $U_i(z)$ and $V_k(z)$ from equations (2.11) the stresses $\sigma = \sigma(z,s)$ and $\tau = \tau(z,s)$ at any point of the cross-sections $z$ = constant will be found according to equations (2.2) and (2.3) also with an accuracy up to $2(m + n)$ arbitrary constants. The stresses $\sigma$ and $\tau$ at the section $z$ = constant for chosen polygonal functions $\varphi_i(s)$ and $\psi_k(s)$ can be expressed in terms of $m + n$ independent generalized static magnitudes. These magnitudes we shall introduce by generalizing the fundamental concepts of the elementary theorem of the bending of beams as was done in our work on the general theory of the thin-walled elastic rods based on the law of sectorial areas (reference 3).

Starting from the idea of the virtural work of the normal and shearing forces $\sigma\delta$ and $\tau\delta$ of the cross-section $z$ = constant on each of the $m + n$ possible displacements of the points of this section in space we introduce the following magnitudes

$$P_j(z) = \int \sigma\varphi_j \, dF \qquad (j = 1,2,\ldots, m)$$

$$Q_h(z) = \int \tau\psi_h \, dF \qquad (h = 1,2,\ldots, n) \tag{3.1}$$

where the contour integrals are taken over the entire cross-sectional element of the shell. The magnitudes $P_j$, $Q_h$ represent generalized longitudinal and transverse forces of the section $z$ = constant. Considering these magnitudes as the internal forces of the shell we express them in terms of the fundamental functions $U_i$ and $V_k$. On the basis of equations (2.2), (2.3), and (3.1) we have

$$\left.\begin{aligned} P_j &= E\sum_i a_{ji}U_i' \\ Q_h &= G\sum_i c_{hi}U_i + \sum_k r_{hk}V_k' \end{aligned}\right\} \quad \begin{pmatrix} i,j = 1,2,\ldots,\ m \\ h,k = 1,2,\ldots,\ n \end{pmatrix} \tag{3.2}$$

If the functions $\varphi_i$ and $\psi_k$ are orthogonal

$$P_j = Ea_{jj}U_j', \quad Q_h = G\left(\sum_j c_{hj}U_j + r_{hh}V_h'\right) \quad (j = 1,2,\ldots,\ m) \tag{3.3}$$

Equations (2.2) can now be written as

$$\sigma = \sum_j \frac{P_j}{a_{jj}}\varphi_j \quad (j = 1,2,\ldots,\ m) \tag{3.4}$$

or in expanded form

$$\sigma(z,s) = \frac{P_1(z)}{a_{11}}\varphi_1(s) + \frac{P_2(z)}{a_{22}}\varphi_2(s) + \ldots + \frac{P_m(z)}{a_{mm}}\varphi_m(s) \tag{3.5}$$

Equation (3.4) represents the generalization of the three-term formula proposed by us for the normal stresses of open cylindrical and prismatic shells (for thin-walled rods with open profile sections) undergoing simultaneous tension, bending in two planes and torsion. The theory of such shells is based on the hypothesis of the nondeformability of the contour of the cross-section.

For the open shell with rigid cross-sectional contour setting

$$\varphi_1 = 1, \quad \varphi_2 = x(s), \quad \varphi_3 = y(s), \quad \varphi_4 = \omega(s) \tag{3.6}$$

where $x(s)$ and $y(s)$ are the cartesian coordinates of an arbitrary point of the contour in the principal central axes of the section and $\omega(s)$ is twice the area of the sector determined by the arc $M_oM$

and the straight lines joining the ends of this arc with the center of flexure (fig. 4) we obtain for the generalized forces $P_1,\ldots, P_4$, the first four terms of the series (3.4), the following values (reference 3, p. 48).

$$
\begin{aligned}
P_1 &= \int \sigma 1 \, dF = N, \quad P_2 = \int \sigma x \, dF = M_y \\
P_3 &= \int \sigma y \, dF = M_x, \quad P_4 = \int \sigma \omega \, dF = B
\end{aligned}
\tag{3.7}
$$

The first three equations of (3.7) determine the known statical magnitudes (normal force and moments) of the cross-section of a beam; the fourth equation determines a new static magnitude having the dimensions kilograms per centimeter$^2$ and representing the work of all the elementary longitudinal forces $\sigma \, dF$ of the shell on the deplanation of the cross-section determined by the law of sectorial areas. This magnitude is termed by us the bimoment. The condition of orthogonality of the four fundamental functions (3.6) is determined by us in the form

$$
\int 1x \, dF = \int 1y \, dF = \int xy \, dF = 0, \quad \int 1\omega \, dF = \int x\omega \, dF = \int y\omega \, dF = 0 \tag{3.8}
$$

Of these conditions the first three coincide with those known from the theory of the bending of beams and which determine the principal central axes of the cross-section of the shell. The second group of equations (3.8) refers to the sectorial geometric characteristics of the cross-section of the shell and determines the sectorial origin $M_o$ (the starting point for computing the sectorial area) and the center of flexure of the shell. For the conditions (3.8) the bimoment $B = B(z)$ represents a generalization statically equivalent to the vanishing of the longitudinal force.

Corresponding to equations (3.6) and (3.8) we obtain for the geometric characteristics $a_{jj}$ $(j = 1,2,3,4)$ of the first four terms of the series (3.5) the values

$$
\begin{aligned}
a_{11} &= \int 1^2 \, dF = F, \quad a_{22} = \int x^2 \, dF = J_y \\
a_{33} &= \int y^2 \, dF = J_x, \quad a_{44} = \int \omega^2 \, dF = J_\omega
\end{aligned}
\tag{3.9}
$$

The first three of these characteristics agree with the well-known fundamental characteristics of the cross-section (area and moment of inertia) in the theory of the bending of a beam. The fourth characteristic represents a new magnitude having the dimensions of centimeters$^6$ and denoted by us, in analogogy with $J_x$, $J_y$, the sectorial moment of inertia.

The equation (3.5) for the thin-walled open shell, keeping only the first four terms, assumes on the basis of equations (3.7) and (3.9) the form

$$\sigma = \frac{N}{F} + \frac{M_y}{J_y} x + \frac{M_x}{J_x} y + \frac{B}{J_\omega} \omega \tag{3.10}$$

The first three terms of this equation determine the stresses of the thin-walled rod in the case of tension (compression) and bending. The fourth term determines the stress arising in the case of torsion of the rod and distributed over the section by the law of sectorial areas $\omega = \omega(s)$. The static magnitudes $N = N(z)$, $M_x = M_x(z)$, and $M_y = M_y(z)$ are found by the well-known methods of the resistance of materials. The static magnitude $B = B(z)$ refers to the normal stress $\sigma_\omega = (B/J_\omega)_\omega$ due to the torsion determined from the differential equation

$$B'' - \frac{GJ_d}{EJ_\omega} B + \frac{m}{EJ_\omega} = 0 \tag{3.11}$$

in which $GJ_d$ is the stiffness of the rod in pure torsion determined by the theory of Saint Venant and $m = m(z)$ is the external torsional moment relative to the center of flexure.

From the analogy here given it follows that equation (3.5) for the shell with closed multiply connected deformable contour for a choice of the first three functions $\varphi_j(s)$ by the equations $\varphi_1 = 1$, $\varphi_2 = x(s)$ and $\varphi_3 = y(s)$ and the conditions of orthogonality (2.17) can be represented in the form

$$\sigma = \frac{N}{F} + \frac{M_y}{J_y} x + \frac{M_x}{J_x} y + \frac{P_4}{a_{44}} \varphi_4 + \frac{P_5}{a_{55}} \varphi_5 + \ldots + \frac{P_m}{a_{mm}} \varphi_m \tag{3.12}$$

where the first three terms refer to the stresses distributed over the section according to the law of plane deformation and the corresponding elementary theory of the bending of beams. The remaining generalized longitudinal forces $P_4(z)$, $P_5(z)$, ..., $P_m(z)$ for the contour of the shell possessing in the longitudinal direction a number of degrees of freedom greater than three will represent the internal "longitudinal" forces having essentially the same character as the forces $N$, $M_x$, and $M_y$ of a beam, the only difference being that these forces, for a choice of an orthogonal system of functions $\varphi_i(s)$, each represent over the cross-section a balanced system of longitudinal forces and arise as a result of the deplanation of the section. In contrast to the longitudinal force $N$

and the bending moments $M_x$ and $M_y$ the generalized longitudinal forces $P_4, P_5, \ldots, P_m$ associated with the deplanation of the section will be denoted as the longitudinal bimoments. These bimoments correspond to the generalized coordinates $\varphi_4, \varphi_5, \ldots, \varphi_m$ of the deplanation of the cross-section.

The geometric characteristics

$$a_{jj} = \int \varphi_j^2 \, dF \tag{3.13}$$

by analogy with the well-known magnitudes

$$a_{22} = \int x^2 \, dF = J_y, \qquad a_{33} = \int y^2 \, dF = J_x \tag{3.14}$$

are denoted as the longitudinal bimoments of inertia.

In a similar manner the physical sense of the generalized transverse forces $Q_h(z)$ $(h = 1,2,\ldots n)$ determined by the corresponding equations (3.1) and (3.3) can be explained.

If of the $n$ generalized transverse displacements we give the first three functions $V_1(z)$, $V_2(z)$, and $V_3(z)$ the sense of three independent displacements of the elementary lamina as a rigid system in the plane of the cross-section of the shell and set

$$\psi_1(s) = x'(s), \quad \psi_2(s) = y'(s), \quad \psi_3(s) = x(s)y'(s) - y(s)x'(s) \tag{3.15}$$

the magnitudes $Q_1(z)$, $Q_2(z)$, and $Q_3(z)$ corresponding to the coordinates of the displacements will represent the first two transverse forces and the torsional moment of the entire section $z$ = constant, respectively. The remaining static magnitudes $Q_4(z)$, $Q_5(z)$, and $Q_n(z)$ will represent the generalized transverse forces each computed as the work of the elementary shearing forces $\tau \, dF$ by the corresponding contour (tangential) displacements, that is by the displacements of the shell determined by the generalized coordinates $\psi_4(s)$, $\psi_5(s), \ldots, \psi_n(s)$ of the deformations of the contour of the cross-section.

In contrast to the forces $Q_1 = Q_x$ and $Q_2 = Q_y$ and the torsional moment $Q_3 = H$, the generalized forces $Q_4, Q_5, \ldots, Q_n$ corresponding to the components $V_4, V_5, \ldots, V_n$ of the deformations of the contour of

the section we shall denote as the transverse bimoments of the shell. Equations (3.3) determine the relation between the generalized forces and the generalized displacements of the shell.

The generalized forces $P_j(z)$ $(j = 1,2,\ldots, m)$, $Q_h(z)$ $(h = 1,2,\ldots, n)$ characterizing the state of the normal and shearing forces in the cross-section of the shell are likewise determined with an accuracy up to $2(m + n)$ constants of integration of the system of differential equations of the shell. We now assume that any edge of the shell $z = z_o$ is acted upon by a given system of normal and shearing forces. Let $p^o(z_o,s)$, $q^o(z_o,s)$ represent the corresponding normal and shearing forces referred to unit length of the contour at the point $s$. These forces as functions of $s$ may be given entirely arbitrarily. In our problem the longitudinal and transferse forces at any section $z$ = constant are determined by a finite number of independent static magnitudes of the corresponding finite number of degrees of freedom of the displacements of the contour line around the section and in its plane. These magnitudes are generalized longitudinal and transverse forces which for the given normal and shearing forces $p^o(z_o,s)$, $q^o(z_o,s)$ are determined by the equations

$$P_j{}^o = \int p^o \varphi_j \, ds, \quad Q_h{}^o = \int q^o \psi_h \, ds \qquad (j = 1,2,\ldots, m, \ h = 1,2,\ldots, n) \tag{3.16}$$

The above equations together with equations (3.2) lead, for the section $z = z_o$, to the equations

$$Ea_{ji}U_i' = \int p^o \varphi_j \, ds, \qquad G(c_{hi}U_i + r_{hk}U_k') = \int q^o \psi_h \, ds \tag{3.17}$$

These equations determine the relation between the required generalized displacements and the given generalized forces at the bounding section $z = z_o$.

Having the general integral of the differential equations of the shell and making use of the general formulas (3.17) we can determine the state of the deformations and stresses of the shell for the most varied boundary conditions on the sections $z = 0$ and $z = l$ given in terms of stresses, displacements, or partly stresses and partly displacements.

## 4. Shells Having a Single Degree of Freedom for Deplanation of the Section and Deformation of the Contour

Let us consider the class of shells for which the deplanation of the section and the deformation of the contour are each determined by a single parameter which is a function of the coordinate $z$. To such

shells may be reduced a number of practically important problems encountered in the construction of thin-walled structures in various applied fields of technology.

As a very simple example let us consider the shell, the cross-section of which is represented in figure 5. We assume that the vertical plates of the shell have cylindrical hinges at the lower longitudinal edges and are fixed over the entire supporting length against longitudinal displacements. For such a shell the elementary transverse lamina as a plane hinged system with the supporting points completely fixed, possesses three degrees of freedom, two out of its plane and one in its plane. We assume further that the section of the shell is symmetrical relative to the middle vertical line and that the shell is acted upon by a transverse load of given intensity $q = q(z)$ applied in the plane of the horizontal plate. In this case the longitudinal displacements of the shell $u(z,s)$ in the section $s$ = constant for any two points of the section symmetrically located with respect to the axis of symmetry will be equal in absolute value and opposite in sign.

These displacements are determined by a single parameter $U_1(z)$ for which we take the longitudinal displacement (taking it positive) of the upper right angle of the section. The graph of the displacements represented for $U_1{}^* = 1$ as the function $\varphi_1(s)$ is shown in figure 6. The law of distribution of the longitudinal displacements and therefore also the normal stresses over the section $z$ = constant is postulated by the displacements of the points determined by this function. This law differs from that of plane sections.

The derivative $\varphi_1{}'(s)$ within the limits of each of the segments of the section remains constant and in absolute value equal to $1/d_1$ for the vertical portion and $2/d_2$ for the horizontal portion where $d_1$ and $d_2$ are the lengths of the vertical and horizontal portions.

The transverse contour displacements of the shell for the case where the longitudinal edges are fixed, are determined likewise by a single parameter $V_1(z)$. For this parameter we take the bending of the horizontal plate in the section $z$ = constant (fig. 7). The contour axial displacement of the points of the section of the shell that are determined by the function $\psi_1(s)$ corresponding to unit value of the required magnitude $V_1{}^* = 1$ is shown in figure 8. These displacements on the vertical portions of the section are equal to zero and on the horizontal equal to unity. The arrow in figure 8 gives the direction of the positive displacement. Figure 9 shows the graph of the bending moments of the elementary transverse frame corresponding to the displacement $V_1{}^* = 1$ of the horizontal elements.

For the moment $M_1$ at the joint of the frame it is not difficult by the methods of structural mechanics to obtain the formula

$$M_1 = \frac{6E}{2d_1^2 | J_1 + d_1 d_2 | J_2} V_1 \tag{4.1}$$

where $d_1$ and $d_2$ are the widths of the vertical and horizontal plates, $J_1$ and $J_2$, the vertical and horizontal moments of inertia of the elementary frame.

In the problem under consideration there are thus to be determined two fundamental functions, the longitudinal displacement $U_1(z)$ determining the deplanation of the section and the transverse displacement $V_1(z)$ determining the deformation of the contour of the section.

Setting in equations (2.11) $i$, $j = 1$; $h,k = 1$ we obtain

$$\gamma a_{11} U_1'' - b_{11} U_1 - c_{11} V_1' = 0, \quad c_{11} U_1' + r_{11} V_1'' - \gamma s_{11} V_1 + \frac{1}{G} q_1 = 0 \tag{4.2}$$

The coefficients of equation (4.2) are computed by formulas (2.13). Applying these formulas and making use of the graphs of the corresponding functions given in figures 6, 8, and 9 we obtain

$$a_{11} = \int \varphi_1^2 \, dF = \frac{1}{3}(2F_1 + F_2), \qquad b_{11} = \int \varphi_1'^2 \, dF = 2\left(\frac{\delta_1}{d_1} + 2\frac{\delta_2}{d_2}\right)$$

$$c_{11} = \int \varphi_1' \psi_1 \, dF = 2\delta_2, \qquad r_{11} = \int \psi_1^2 \, dF = F_2 \tag{4.3}$$

$$s_{11} = \int \frac{M_1^2 \, ds}{J} = \frac{12}{d_1^2 \left(2d_1 | J_1 + d_2 | J_2\right)}$$

where $\delta_1$ and $\delta_2$ are the thickness of the vertical and horizontal plates respectively, $F_1$ and $F_2$, the cross-section areas of these plates and $J_1$ and $J_2$, the moments of inertia referred to unit length of the longitudinal sections of the shell on the vertical and horizontal parts, respectively:

$$F_1 = d_1 \delta_1, \qquad F_2 = d_2 \delta_2 \tag{4.4}$$

$$J_1 = \frac{\delta_1^3}{12}, \qquad J_2 = \frac{\delta_2^3}{12} \tag{4.5}$$

If the shell has along its length a number of transverse frames the moments of inertia $J_1$ and $J_2$ must be computed with account taken of the resistance of these frames to bending.

Equations (4.2) for the coefficients (4.3) will be completely determinate for a given shell. By giving different values to the magnitudes $d_1$, $d_2$, $\delta_1$, $\delta_2$, $J_1$, and $J_2$ a number of special problems can be considered referring to the computation of a given shell for an antisymmetric load on the cross-section. Leaving in equations (4.2) the coefficients $a_{11}, b_{11}, \ldots, s_{11}$ as yet arbitrary let us consider more in detail the solution of these equations and incidentally explain the physical side of the elastic computation model.

The system of differential equations (4.2) by eliminating the longitudinal displacement reduces to a single equation relative to the transverse displacement of bending of the horizontal plate $V_1(z)$. This equation may be written in the form

$$V_1^{IV} - 2A^2V_1'' + B^4V_1 + \frac{1}{c_{11}G}(b_{11}q - \gamma a_{11}q'') = 0 \qquad (4.6)$$

where

$$A^2 = \frac{1}{4a_{11}r_{11}}(\gamma^2 a_{11}s_{11} + b_{11}r_{11} - c_{11}^2), \quad B^4 = \frac{b_{11}s_{11}}{a_{11}r_{11}}; \quad \gamma = \frac{E}{G} \qquad (4.7)$$

Differential equation (4.6) expresses the equilibrium of the elementary transverse frame in the dense elastic medium which is constituted by the given shell relative to this frame. In its form this equation agrees with the equation of the bending of a beam on an elastic support resisting not only bending but also shear along the line of contact of the beam with the support. The elastic medium in equation (4.6) is represented by two independent magnitudes $A^2$ and $B^4$, which may be called the generalized elastic characteristics of the beam, each having a single degree of freedom for the deplanation of the section and the deformation of the contour. These characteristics, as is seen from equations (4.7) and (4.3) are determined by the functions $\varphi_1(s)$ and $\psi_1(s)$ referring respectively to the deplanation and deformation of the contour of the shell and by the form and geometric dimensions of the cross-section of the shell, by the ratio $\gamma = \frac{E}{G}$ of the elasticity modulus and the stiffness in bending in the transverse direction represented by the moments of inertia $J_1$ and $J_2$ of the elementary frame of width $dz = 1$. Thus, for example, for the shell represented in figure 5 the elastic characteristics $A^2$ and $B^4$ for given dimensions of this shell on the basis of formulas (4.3) assume entirely definite values.

The characteristic $B^4$ as is seen from the second of formulas (4.7) is proportional to the coefficient $s_{11}$ referring to the bending of the elementary transverse lamina as a frame. If the plates of the shell are hinge connected at the joints, the transverse bending moments of the elementary lamina on changing its shape is equal to zero. The coefficient $s_{11}$ and therefore also the characteristic $B^4$ in this case becomes zero:

$$s_{11} = B^4 = 0 \tag{4.8}$$

and equation (4.6) for the zero moment shell goes over into the simpler equation

$$V_1{}^{IV} - 2A^2V_1{}'' + \frac{1}{c_{11}G}(b_{11}q - \gamma a_{11}q'') = 0 \tag{4.9}$$

in which now

$$A^2 = \frac{b_{11}r_{11} - c_{11}{}^2}{4a_{11}r_{11}} \tag{4.10}$$

Proceeding to the solution of the differential equations of the class of shells here considered we note that the two kinematic magnitudes $U_1(z)$ and $V_1(z)$ by formulas (3.2) are reduced to two kinematic magnitudes corresponding to the static magnitudes $P_1(z)$ and $Q_1(z)$ introduced by us, the first of which represents the required longitudinal force generalized for the deplanation of the section (longitudinal bimoment) and the second the generalized transverse force in deformation of the contour (the transverse bimoment). We thus have four required magnitudes $U_1(z)$, $V_1(z)$, $P_1(z)$, and $Q_1(z)$ representing generalizations of four magnitudes that are well known in the theory of the resistance of materials on the bending of beams: the deflection $y(z)$, the angle of rotation of the section $y'(z)$, the bending moment $M(z)$, and the transverse force $Q(z)$.

The arbitary constants of the integration are determined by the boundary conditions which, depending on the type of problem may be given in generalized displacements $U_1$ and $V_1$, in generalized forces $P_1$ and $Q_1$, or partly in displacements and partly in forces, only two conditions being required at each of the bounding sections of the shell $z = 0$, $z = l$. Thus for example if the infinitely long shell with section represented in figure 10 is acted upon by a concentrated transverse horizontal force $P$ the conditions for determining the constants of integration on the part $0 \leqslant z \leqslant \infty$ assume the form

$$U_1 = 0, \quad G(c_{11}U_1 + r_{11}V_1') = -\frac{P}{2} \quad \text{for} \quad z = 0$$

$$V_1 = 0, \quad G(c_{11}U_1 + r_{11}V_1') = 0 \quad \text{for} \quad z = \infty \tag{4.11}$$

The first and third conditions are kinematic and express the fact that in the plane of action of the force $P$ the deplanation of the section is equal to zero (section $z = 0$ by symmetry remains plane) and at infinity the deformation of the contour becomes zero. The second and fourth conditions are static and refer to the generalized (in the sense of virtual work) transverse force $Q$ (the transverse bimoment) which in the section $z = 0$ must be in equilibrium with the external transverse load and at infinity becomes zero.

In table 3 are given the values of the generalized displacements for various sections of the infinitely long shell, represented on figure 10 and loaded by a transverse horizontal force P. These displacements and stresses are computed for shells consisting of three plates of equal width and thickness for a ratio of thickness to width of plate equal to 0.01.

In the first row of table 3 are given the values of the relative coordinate $\zeta = \frac{z}{d}$ (d is the width of a plate), in the second row are given the magnitudes proportional to the longitudinal displacements $U_1$ of the right joint of the "frame" of the cross-section; the third row refers to the transverse displacements $V_1$ (deflection of the horizontal plate), determining the deformation of the contour of the section of the shell. From these displacements are determined the transverse bending moments of the shell arising from the stiffness of the joint of the plates (horizontal and vertical). The maximum bending moments, according to the graph shown in figure 9, arise at the joints of the transverse frame and are determined by the formula

$$M(\zeta) = \pm \frac{2EJ}{d^2} V_1(\zeta)$$

in which the plus sign refers to the left and the minus sign to the right joint. In the fourth row of the table are given the magnitudes proportional to the longitudinal normal stresses $\sigma = \sigma(\zeta)$ referring to the points of juncture of the cross-section. The graph of these stresses over the section $z$ = constant agrees with the graph of the deplanation of the section, shown in figure 6. The last row refers to the tangential stresses $\tau = \tau(\zeta)$ forming in the section $z$ = constant the flow of shearing forces $s = \tau\delta$.

The values of the displacements and stresses given in table 3 refer to the sections of the shell with positive abscissas $\zeta = \frac{z}{d}$ = constant. For sections with negative abscissas the magnitudes $V_1$ and $\sigma$ retain their values while $U_1$ and $\tau$ reverse their signs. It is seen from table 3 that the deflections $V_1$ of the horizontal plate and therefore the transverse bending moments proportional to the deflections (the graph of these moments as a function of $\zeta$ is shown in fig. 9) decrease at a slower rate with increase in distance from the plane of action of the force than the stresses $\sigma$.

## 5. Prismatic Shell with Closed Rectangular Section

As a second example we consider a shell having in the cross-section a closed rectangular contour with two axes of symmetry (fig. 11).

According to the theory developed above, the elementary transverse lamina $dz = 1$ of such a shell possesses $4^\circ$ of freedom both in the plane and about the plane of the cross-section. The deplanation of the section and the deformation of the contour are each represented by a single parameter.

Referring the cross-section of the shell to the principal central axes Ox, Oy the required displacements $u(z,s)$ of an arbitrary point $(z,s)$ of the middle surface is represented in the form

$$u = U_1\varphi_1 + U_2\varphi_2 + U_3\varphi_3 + U_4\varphi_4, \qquad v = V_1\psi_1 + V_2\psi_2 + V_3\psi_3 + V_4\psi_4 \tag{5.1}$$

where

$$\begin{gathered}\varphi_1 = 1, \quad \varphi_2 = x(s), \quad \varphi_3 = y(s), \quad \varphi_4 = x(s)y(s) \\ \psi_1 = h(s), \quad \psi_2 = x'(s), \quad \psi_3 = y'(s), \quad \psi_4 = x'(s)y(s) + x(s)y'(s)\end{gathered} \tag{5.2}$$

where $x(s)$, $y(s)$ are the coordinates of an arbitrary point of the contour determined by the distance $s$ along the contour, $h(s)$ is the length of the perpendicular dropped from the center of the rectangle on the line of the contour; the primes denote derivatives with respect to the parameter $s$.

The magnitudes $U_i(z)(i = 1,2,3,4)$ refer to the generalized longitudinal displacements of the shell and for chosen functions $\varphi_i(s)$ $(i = 1,2,3,4)$ the magnitudes $U_1(z)$ are the displacements of the section $z$ = constant in the direction of the Oz axis, the

magnitudes $U_2(z)$ and $U_3(z)$ are the angles of rotation of the section $z = \text{constant}$ relative to the axes Oy, Ox, respectively, and the magnitude $U_4$ is the generalized deplanation of the section. The magnitudes $V_k(z)$ $(k = 1,2,3,4)$ represent: $V_1(z)$ – the angle of rotation of the section relative to the axis Oz (angle of torsion), $V_2(z)$, $V_3(z)$ – the deflections in the direction of the axes Ox and Oy, and $V_4(z)$ – the generalized deformation of the contour of the section.

It is easily shown that for chosen functions $\varphi_i(s)$ and $\psi_k(s)$ the system of differential equations (2.16) breaks down into four independent systems of equations. Of these systems three refer to tension (compression) and bending of the shell as a beam following the hypothesis of plane sections. This hypothesis in the first of formulas (5.1) is represented by the sum of the first three terms. The fourth system of equations consists of three simultaneous differential equations in the functions $U_4(z)$, $V_1(z)$, and $V_4(z)$. Leaving out in our problem the case of tension and bending we consider the equilibrium of the shell in the case where it is acted on by external forces giving rise to torsion, deplanation of the section and deformation of the contour (fig. 12(b)).

In figure 13 is given the diagram of the function $\varphi_4(s) = x(s)y(s)$ characterizing the law of variation of the longitudinal displacements for a unit deplanation $V_4^* = 1$. This function represents a generalization of the law of sectorial areas previously given by us for thin-walled open rods. Figures 14 and 15 give the diagrams of the functions $\psi_4 = \varphi_4'(s)$, $\psi_1 = h(s)$. The signs of the magnitudes $\psi_1$ and $\psi_4$ on these diagrams are denoted by arrows. The letters denote the length of the sides of the rectangle (fig. 11). Figure 16 gives the diagram of the transverse bending moments corresponding to the generalized deformation of the contour, taken equal to one. This diagram as well as the diagram of unit deplanation shown in figure 13 is antisymmetric relative to both axes of symmetry of the section. The maximum bending moment arising at the juncture point of the contour for unit generalized deformation $V_4^* = 1$ is determined by the formula

$$M = \frac{12}{d_1 \,\big|\, (EJ_1) + d_2 \,\big|\, (EJ_2)} \tag{5.3}$$

in which $J_1$ and $J_2$ are the moments of inertia of the vertical and horizontal elements of the closed rectangular frame of width $d_1$, $d_2$. If the shell consists of only rectangular plates having stiff junctures at the ribs $J_1 = \frac{\delta_1^3}{12}$, $J_2 = \frac{\delta_2^3}{12}$ and therefore

$$M = \frac{E}{d_1 \Big| \delta_1{}^3 + d_2 \Big| \delta_2{}^3} \tag{5.4}$$

Expanding equations (2.11) for the case of torsion of the shell, computing the coefficients of these equations by formulas (2.13), and using the diagrams given in figures 13, 14, and 15, we obtain

$$\frac{1}{24} Ed_1{}^2 d_2{}^2(F_1 + F_2)U_4'' - \frac{1}{2} G\left(d_2{}^2F_2 + d_2{}^2F_1\right)U_4 - \frac{1}{2} G\left(d_1{}^2F_2 - d_2{}^2F_1\right)V_1'$$

$$- \frac{1}{2} G(d_1{}^2F_2 + d_2{}^2F_1)V_4' + p_4 = 0$$

$$\frac{1}{2} G\left(d_1{}^2F_2 - d_2{}^2F_1\right)U_4' + \frac{1}{2} G\left(d_1{}^2F_2 + d_2{}^2F_1\right)V_1''$$

$$+ \frac{1}{2} G\left(d_1{}^2F_2 - d_2{}^2F_1\right)V_4'' + q_1 = 0 \tag{5.5}$$

$$\frac{1}{2} G\left(d_1{}^2F_2 + d_2{}^2F_1\right)U_4' + \frac{1}{2} G\left(d_1{}^2F_2 - d_2{}^2F_1\right)V_1'' + \frac{1}{2} G\left(d_1{}^2F_2 + d_2{}^2F_1\right)V_4''$$

$$- \frac{96}{d_1 \Big|(EJ_1) + d_2 \Big|(EJ_2)} V_4 + q_4 = 0$$

where $d_1$, $d_2$ are the widths of the vertical and horizontal plates, $F_1$, $F_2$, the cross-section areas of these plates, $p_4 = p_4(z)$, $q_1 = q_1(z)$, $q_4 = q_4(z)$, the generalized external forces referring to unit length of the shell and computed by the formulas

$$p_4 = \int p(z,s)\varphi_4(s)\, ds, \qquad q_1 = \int q(z,s)\psi_1(s)\, ds, \qquad q_4 = \int q(z,s)\psi_4(s)\, ds \tag{5.6}$$

in which $p(z,s)$ and $q(z,s)$ are the loads referred to unit area of the middle surface of the shell and acting in the direction of the generator and contour line respectively.

If the shell is acted upon by a transverse vertical load $q(z)$ applied in the plane of some vertical plate (fig. 12(a)) we have

$$p_1 = 0, \qquad q_1 = \frac{1}{2} qd_2, \qquad q_4 = -\frac{1}{2} qd_2 \tag{5.7}$$

The equations (5.5) may be represented in another form, by taking for the unknown function the longitudinal normal stress $\sigma = \sigma(z)$ and the transverse bending moment $M = M(z)$ referring to the juncture point of the contour. We have

$$U_4' = \frac{4}{d_1 d_2} \frac{\sigma}{E}, \qquad V_4 = \frac{1}{12}\left(\frac{d_1}{EJ_1} + \frac{d_2}{EJ_2}\right) M_4 \tag{5.8}$$

Eliminating from equation (5.5) $U_4$ and $V_4$ with the aid of formulas (5.8) and denoting the angle of torsion of $V_1(z)$ by $\theta(z)$, for $p_1 = 0$, $q_1 = -q_4 = \frac{qd_2}{2}$, we obtain

$$\frac{F_1 + F_2}{6} \sigma'' - \frac{8M}{d_1 d_2} + \frac{1}{2d_1} q = 0$$

$$\frac{8}{d_1 d_2} \sigma + \frac{1}{6}\left(\frac{d_1}{J_1} + \frac{d_2}{J_2}\right) M'' - 8\left(\frac{E}{Gd_1^2 F_2} + \frac{E}{Gd_2^2 F_1}\right) M - \frac{E}{G} \frac{q}{d_2 F_1} = 0 \tag{5.9}$$

$$\theta'' - 4\left(\frac{1}{Gd_1^2 F_2} - \frac{1}{Gd_2^2 F_1}\right) M + \frac{1}{2GF_1 d_2} q = 0$$

The first two of these equations form a symmetric system relative to the functions $\sigma = \sigma(z)$ and $M = M(z)$ and in the case of the homogeneous problem reduce to an equation of the fourth order for $\sigma$ and $M$

$$\sigma^{IV} - 2A^2 \sigma'' + B^4 \sigma = 0, \qquad M^{IV} - 2A^2 M'' + B^4 M = 0 \tag{5.10}$$

or to an equation of the sixth order for the angle of torsion $\theta(z)$

$$\theta^{VI} - 2A^2 \theta^{IV} + B^4 \theta'' = 0 \tag{5.11}$$

where A and B are the generalized elastic characteristics determined by the formulas

$$A^2 = \frac{24}{d_1 \Big| (EJ_1) + d_2 \Big| (EJ_2)} \left( \frac{1}{Gd_2{}^2F_1} + \frac{1}{Gd_1{}^2F_2} \right) \tag{5.12}$$

$$B^4 = \frac{384}{d_1{}^2 d_2{}^2 (F_1 + F_2)} \, \frac{1}{d_1 \Big| J_1 + d_2 \Big| J_2}$$

Equation (5.10) is the fundamental differential equation of the shell here considered having $1^{\circ}$ of freedom for deplanation of the section and one for deformation of the contour. By determining from this equation the function $\sigma(z)$ and then by equations (5.9) the remaining functions $M(z)$, $\theta(z)$, adding to the integrals of the homogeneous equations the particular solutions of the nonhomogeneous system (5.9) we obtain the general integral for the problem here considered. The general solution will be found with an accuracy up to 6 independent arbitrary constants. For determining the latter it is necessary in each particular case to use the boundary conditions which in the given problem will consist of three conditions on each of the ends of the shell. Thus for example, in the case of a shell hinged at the fixed ends $z = 0$, $z = l$ the boundary conditions have the following form

$$\sigma = M = \theta = 0 \quad \text{for} \quad z = 0; \qquad \sigma = M = \theta = 0 \quad \text{for} \quad z = l \tag{5.13}$$

Writing out these conditions we obtain six equations with six arbitrary constants of integration of the system (5.9).

We may remark that in addition to the accurate solution here described, equations (5.9) for the boundary conditions (5.13) can be integrated with the aid of trigonometric series. Bearing in mind conditions (5.13) the solution of equations (5.9) for $q$ = constant may be written in the form

$$\sigma = \sum_{n=1,2\ldots} \sigma_n \sin \frac{(2n-1)\pi z}{l}$$

$$M = \sum_{n=1,2\ldots} M_n \sin \frac{(2n-1)\pi z}{l} \tag{5.14}$$

$$\theta = \sum_{n=1,2\ldots} \theta_n \sin \frac{(2n-1)\pi z}{l}$$

where $\sigma_n$, $M_n$, and $\theta_n$ are the required expansion coefficients of equation (5.14). Substituting (5.14) in (5.9) we obtain

$$\frac{F_1 + F_2}{6} \frac{(2n-1)^2\pi^2}{l^2} \sigma_n + \frac{8}{d_1 d_2} M_n + \frac{2q}{\pi d_1 (2n-1)} = 0$$

$$\frac{8}{d_1 d_2} \sigma_n - \frac{1}{6}\left(\frac{d_1}{J_1} + \frac{d_2}{J_2}\right) \frac{(2n-1)^2\pi^2}{l^2} M_n - 8\left(\frac{E}{G{d_1}^2 F_2} + \frac{E}{G{d_2}^2 F_1}\right) M_n$$

$$- \frac{4E}{G d_2 F_1} \frac{q}{\pi(2n-1)} = 0 \tag{5.15}$$

$$\frac{(2n-1)^2\pi^2}{l^2} \theta_n + 4\left(\frac{1}{G{d_1}^2 F_2} - \frac{1}{G{d_2}^2 F_1}\right) M_n - \frac{2q}{G F_1 d_2 \pi (2n-1)} = 0 \quad (n = 1,2,3,\ldots)$$

From these equations and series (5.14) the required functions $\sigma(z)$, $M(z)$, and $\theta(z)$ may be determined with any initially given degree of accuracy in the case where the shell is acted upon by a vertical uniformly distributed load $q$ in the plane of any vertical plate (fig. 12). We may remark that the series (5.14) for equations (5.15) possess very good convergence and for practical purposes may be restricted to the two or three first terms of the expansion, equation (5.14).

Equations (5.9) represent a particular case of the general eight term equations proposed by the author as early as 1931 (references 2, 3, 4). These equations, thanks to the independent characteristics (elastic E, G, geometric $d_1$, $d_2$, $F_1$, $F_2$, $J_1$, $J_2$), permit considering practically important problems on the computation of thin-walled aeronautical and structural shells of closed rectangular cross-section. Thus, for example, if the shell is reinforced by ribs or diaphragms then for a sufficient stiffness of the transverse frames we can set $EJ_1 = EJ_2 = \infty$; equations (5.9) then go over into the equations of a shell with rigid (nondeformable) contour.

In the case of a shell consisting only of rectangular thin plates hinge-connected along the ribs, the second of equations (5.9), which represents according to the general theory given in the works of the author (references 3 and 4) the equation of continuity of the angular deformations on the contour line of the cross-section, drops out. In the remaining two equations the moments $M$ must be set equal to zero. We obtain two equations for $\sigma$ and $\theta$ referring to the shell with zero moment.

If in equations (5.9) the elastic generalized characteristics $GF_1$ and $GF_2$ referring to the deformations of shear are set equal to infinity, we shall have the case considered in detail in the work of the author (reference 3) and referring to a shell having the properties of deforming with longitudinal tension (compression) of the fibers of the shell and with bending of the elementary closed frame of the cross-section. From this short analysis the meaning of the generalized elastic characteristics $A^2$, $B^4$ of one of the equations (5.10), (5.11) is explained. It is clear that in the case of the loading of the shell by a nonsymmetrical load the stresses and deformations determined on the basis of the solution here described must be combined corresponding to the stresses and deformations arising as a result of bending of the shell as a beam. The elementary theory of the bending of beams, as is seen from the above discussion, constitutes a particular case of the general theory here given.

## 6. Example of the Computation of a Shell of Composite Section

We consider a shell with the section shown in figure 18. We assume that this shell is attached in some manner to the supports $z = 0$,

$z = l$ and is acted upon by a single transverse load, perpendicular tc the axis of symmetry of the section. Of the required generalized displacements in this case of loading two will be longitudinal, $U_1$, $U_2$, and three transverse, $V_1$, $V_2$, $V_3$. Figure 19 shows the antisymmetrical diagrams of the orthogonal functions $\varphi_1$, $\varphi_2$ corresponding to the displacements $U_1$ and $U_2$. Of these functions $\varphi_1$ represents the law of linear distribution of the displacements over the section $z$ = constant. The function $\varphi_2$ represents the deplanation of the section. The generalized longitudinal forces corresponding to the generalized displacements $U_1$ and $U_2$ will be the bending moment about the axis of symmetry and the bimoment.

Figure 20 shows the diagrams of the functions $\psi_1$, $\psi_2$, and $\psi_3$ corresponding to the generalized transverse displacements $V_1$, $V_2$, and $V_3$. The magnitude $V_1$ represents the deflection of the shell in the direction perpendicular to the axis of symmetry, $V_2$ – the angle rotation about the point c on the axis of symmetry, $V_3$ the generalized deformation of the contour of the section. The chosen functions $\psi_1$, $\psi_2$, and $\psi_3$ are orthogonal.

Expanding equations (2.11), computing the coefficients of these equations by formulas (2.13) and using the diagrams given for the functions $\varphi_1$, $\varphi_2$, $\varphi_1'$, $\varphi_2'$, $\psi_1$, $\psi_2$, $\psi_3$ assuming that the plates are hinge–connected to each other (the shell has zero moment) we obtain for the dimensions and loads shown in figure 18 and for $E = 2G$ the equations

$$\left.\begin{aligned}
&\frac{25}{6} Fd^2U_1'' - 4FU_1 + \frac{3F}{25} U_2 - 4FV_2' = 0 \\
&\frac{3F}{25} U_1' + \frac{38}{75} Fd^2U_2'' - \frac{1096}{62} FU_2 + \frac{3}{25} FV_1' - \frac{\sqrt{3}Fd}{8} V_2' + \frac{5\sqrt{3}}{4} FdV_3'' = 0 \\
&4FU_1' - \frac{3}{25} FU_2' + 4FV_1'' = -(P_1 + P_2 + P_3) \frac{1}{G} \\
&\frac{\sqrt{3}}{8} FdU_2' + \frac{33}{16} Fd^2V_2'' = -\left(-\frac{3\sqrt{3}}{8} dP_1 + \frac{\sqrt{3}}{8} dP_2 + \frac{5\sqrt{3}}{8} dP_3\right) \frac{1}{G} \\
&-\frac{5\sqrt{3}}{4} FdU_2' + \frac{25}{4} Fd^2V_3'' = -\left(\frac{5\sqrt{3}}{16} dP_1 - \frac{10\sqrt{3}}{8} dP_2 + \frac{5\sqrt{3}}{8} dP_3\right) \frac{1}{G}
\end{aligned}\right\} \quad (6.1)$$

where $F = d\delta$, $\delta$ — thickness of the plate, $P_1$, $P_2$, and $P_3$ the unit transverse loads. We take $\delta = 0.01$ m, $d = 0.50$ m and $l = 3.00$ m. We assume that the shell is hinge-connected on the supports $z = 0$, $z = l$. Then for $z = 0$, $z = l$

$$U_1' = U_2' = V_1 = V_2 = V_3 = 0 \tag{6.2}$$

For these boundary conditions equations (6.1) can be solved by the "approximate" method in the form of trigonometric series for $U$ in $\cos (n\pi z/l)$ and for $V$ in $\sin (n\pi z/l)$. Restricting ourselves to the first terms of the expansion, assuming $P_1$ = constant, $P_2$ = constant, $P_3$ = constant and solving the system of algebraic equations which are obtained from equations (6.1) on substituting in them

$$U_1 = A_1 \cos \frac{\pi z}{l}, \ldots, \quad V_3 = B_3 \sin \frac{\pi z}{l}, \quad P_1 = \frac{4}{\pi} P_1 \sin \frac{\pi z}{l} \tag{6.3}$$

we shall have

$$\begin{aligned}
U_1 &= \frac{1}{E} (-616P_1 - 213P_2 - 213P_3) \cos \frac{\pi z}{l} \\
U_2 &= \frac{1}{E} (124P_1 - 363P_2 + 126P_3) \cos \frac{\pi z}{l} \\
V_1 &= \frac{1}{E} (527P_1 + 514P_2 + 527P_3) \sin \frac{\pi z}{l} \\
V_2 &= \frac{1}{E} (-318P_1 + 162P_2 + 432P_3) \sin \frac{\pi z}{l} \\
V_3 &= \frac{1}{E} (163P_1 - 536P_2 + 245P_3) \sin \frac{\pi z}{l}
\end{aligned} \tag{6.4}$$

The normal stresses $\sigma(z,s)$ are determined by the formula

$$\sigma = E \frac{\partial u}{\partial z} = E(U_1'\varphi_1 + U_2'\varphi_2) \tag{6.5}$$

where $\varphi_1 = \varphi_1(s)$ and $\varphi_2 = \varphi_2(s)$ are the generalized coordinates represented in figure 19. Substituting the obtained values $U_1 = U_1(z)$ and $U_2 = U_2(z)$ in equation (6.5) we obtain

$$\sigma_1 = (+645P_1 + 223P_2 + 223P_3)\varphi_1(s) \sin \frac{\pi z}{l}$$

$$\sigma_2 = (-130P_1 + 380P_2 - 132P_3)\varphi_2(s) \sin \frac{\pi z}{l} \qquad (6.6)$$

From the first of these equations are determined the normal stresses $\sigma_1$ referring to the bending of the shell as a beam ($\varphi_1$ represents the distance from the axis of symmetry of the section to the point of the contour.) The second of formulas (6.6) refers to the stresses arising as a result of the deplanation of the section.

The maximum stresses $\sigma_1$ and $\sigma_2$ arise in the middle transverse section $\left(\text{for } z = \frac{l}{2}\right)$, $P_1$, $P_2$, $P_3$ are the intensities of the transverse loads in kilograms per meter. The stresses $\sigma$ determined by formulas (6.6) are obtained in kilograms per meter$^2$. By comparing $\sigma_1$, $\sigma_2$ it follows that the stresses $\sigma_2$ due to the bimoment are of the same order as the stresses $\sigma_1$ due to the bending. Thus for example if the shell is acted upon by only a single transverse load $P_2$, then setting in equation (6.6) $P_1 = P_3 = 0$ we obtain $z = \frac{l}{2}$

$$\sigma_1 = 223P_2\varphi_1(s), \qquad \sigma_2 = 380P_2\varphi_2(s) \qquad (6.7)$$

For the lower point $a$ of the intersection of the end vertical plate with the inclined one according to figure 19 we shall have

$$\left.\begin{aligned} \sigma_1{}^a &= +223P_2 d = 223 \times 0.5P_2 = 111.5P_2 \\ \sigma_2{}^a &= -380P_2 \frac{7}{25} d = -380 \times 0.14P_2 = -53.3P_2 \end{aligned}\right\} \text{(in kg/m}^2\text{)} \qquad (6.8)$$

The total stress at this point will be

$$\sigma^a = 58.2P_2 \text{ (in kg/m}^2\text{)}$$

It is seen from this example that the factor of deplanation of the section is of essential significance. We may remark that from the formula (6.6) for $\sigma_2$ it follows that there exists a system of loads $P_1$, $P_2$, $P_3$ for which deplanation of the section is absent. These loads in the given example are evidently connected by the relation

$$-130P_1 + 380P_2 - 132P_3 = 0 \qquad (6.9)$$

## 7. On the Principle of Saint-Venant in the Theory of Shells

Let the shell with the cross-section shown in figure 11 have the dimensions

$$d_1 = d_2 = d, \qquad \delta_1 = \delta_2 = \delta \qquad (7.1)$$

The moments of inertia $J_1$ and $J_2$ in the absence of ribs assume the values

$$J_1 = J_2 = \frac{\delta^3}{12} \qquad (7.2)$$

where $\delta$ is the thickness of the plates of the shell which in the given case we assume the same. Further let $E = 2G$.

We consider an infinitely long shell and assume that at the initial section there is applied a system of longitudinal forces reducing to a bimoment and therefore representing the balanced longitudinal load corresponding to the deplanation of the section. The distribution of the stresses $\sigma$ due to such load statically equivalent to zero over the section $z$ = constant is defined by the antisymmetrical diagram shown in figure 13. We assume further that the system of longitudinal forces is such that the normal stress due to these forces at a juncture point of the cross-section is in absolute value equal to unity. The tangential stresses at the initial section $z$ = constant will be considered equal to zero (fig. 21).

Setting in equations (5.5) $p = 2$, $p = q = 0$, using the dimensions (7.1), integrating these equations and remembering the above boundary conditions at the intial section $z = 0$ and the condition of finiteness at $z = \infty$ we obtain a definite solution for the infinitely long shell of square section loaded at the initial section by a single balanced longitudinal load.

In figure 22 are shown the graphs of the variation of the magnitudes $\frac{u'}{d}$ proportional to the normal stress $\sigma = \sigma(z)$ as a

function of the position of the section along the shell. In figure 23 are given the graphs of the variation of the magnitude $V_4/d^2$ proportional to the transverse bending moment M and characterizing the deformation of the contour of the shell. On these graphs along the axis of abscissas are laid off the relative coordinates $\zeta = \frac{z}{d}$ giving the distance in fractions of the width of the plate d from the origin $z = 0$ to the section $z$ = constant under consideration. On the axis of ordinates are laid off in figure 22 the values $u'/d$ and on figure 23, $V_4/d^2$ computed for different values of the ratio $\delta/d$ of the thickness of the plate to its width.

From the graphs for $\sigma = Eu'$ shown in figure 22, it is seen that for a thickness of shell $\delta = 0.01d$ the normal stresses $\sigma$ due to the balanced longitudinal load do not have the character of local stresses and decrease very slowly with increasing distance from the point of application of this load. Even at sections at a distance $z = 8d$ from the initial section these stresses constituted 20 percent of the given stress at the section $z = 0$. The rate of decrease depends on the ratio $\delta/d$. On increasing this ratio the rate of decrease increases. This increase however does not occur rapidly enough for the stresses determined by the fourth term of formula (5.1) and arising from the deplanation of the section to have the character of local stresses. For the case $\frac{\delta}{d} = 0.1$ the stresses $\sigma$ in the section $z = 2.70d$ still remain appreciable.

From the example given it follows that the principle of Saint-Venant, as already several times noted in our previous works (references 1 and 4), has a restricted field of application in the theory of shells. Thus, for example, for the shell here considered for $\frac{\delta}{d} = 0.01$ and for a length not exceeding 15 to 20 times the width of a single plate the stresses $\sigma$ due to the balanced longitudinal load applied at both ends of this shell, even at the middle cross-section at the greatest distance from the ends of the shell constitute 25 to 30 percent of the given stresses at the end sections $z = 0$, $z = l$. Thus, in practical problems the stresses due to the balanced longitudinal (or transverse) load for thin shells of finite length will not be local stresses. The principle of Saint-Venant applicable in a practical sense to dense bodies can lead, on application to shells, as has been shown in our work on thin-walled rods, to false results. It follows that the elementary theory of the bending of shells as beams represented by the first three terms of the general law (5.1) and based essentially on the principle of Saint-Venant is applicable to particular cases, namely, when the external forces and the boundary conditions are such that the bimoment of the longitudinal forces associated with the deplanation of the section is equal to zero, that is in those cases where the system of equations (5.5) has a zero solution. In the light of our discussion above, the principle of Saint-Venant has a restricted field of application also in the theory of shells. Thus if the closed circular cylindrical shell is subjected at the sections $z = 0$, $z = l$ to normal stresses varying along the section according to the law $\cos^2\theta$ where $\theta$ is the central angle, and

reducing therefore to a balanced longitudinal load, the stresses in the center cross-section $z = \frac{l}{2}$ for a length of shell $l = 20R$ and thickness $\delta = 0.01R$ (R is the radius of the circle) constitute, as shown by the investigations of the author, from 40 to 50 percent of the given stresses at the end sections. With decrease in the thickness $\delta$ of the shell the stresses in the section $z = \frac{l}{2}$ increase. In the case of a shell with zero moment the stresses over the entire length remain constant. In the cylindrical or prismatic shell the stresses due to the balanced longitudinal forces at each of the sections $z = 0$, $z = l$ have the character of local stresses only in the case of a sufficiently thick shell for $\frac{\delta}{R} > \frac{1}{30}$, or if the thin shells for a thickness $\frac{\delta}{R} < \frac{1}{30}$ are reinforced over the length by sufficiently rigid ribs.

The problem brought out above connected with the principle of Saint-Venant can be easily investigated with the aid of equations (5.5) by varying in these equations the magnitudes $J_1$, $J_2$ representing for a ribbed shell mean (referring to unit length) reduced moments of inertia of the longitudinal sections of the shell. With increase in the mean bending stiffnesses $EJ_1$ and $EJ_2$ of the elements of the transverse closed frame the rate of decrease of the stresses due to the longitudinal balanced load increases. For $EJ_1 = EJ_2 = \infty$, that is in the case of a shell with rigid contour, the additional stresses connected with the deviation from the hypothesis of plane sections for longitudinal balanced load will have a local character. This remark refers to shells of closed profile. In the case of shells and thin-walled rods of open profile the stresses due to the bimoment, even in the absence of deformation of the contour of the section, extend over a considerable part of the length of the shell and do not have the character of "local" stresses.

## 8. General Theory of the Vibrations of Prismatic Multiply-Connected Shells

The general differential equations (2.11) for the coefficients of these equations (2.13) and free terms (2.15) refer to the problem of the equilibrium of thin-walled spatial multiply-connected prismatic shell. These equations are generalized also to the theory of the vibration of such shells. For this it is necessary in equations (2.11) to understand by the forces $p_i$ and $q_h$ inertia forces arising from the vibrations of the shell for given forms, with an accuracy up to a certain number of parameters, of the longitudinal and transverse displacements.

Considering the displacements u and v of any point of the surface of the shell as a function of three independent variables z, s, t where t is the time, $\gamma$ is the weight of unit volume of the material, and g the acceleration of gravity, we can write

$$\bar{p} = -\frac{\gamma\delta}{g}\frac{\partial^2 u}{\partial t^2}, \qquad q = -\frac{\gamma\delta}{g}\frac{\partial^2 v}{\partial t^2} \tag{8.1}$$

From the above formulas are determined the surface inertia forces, that is the forces referred to unit surface area. These forces act as follows: $\bar{p}$ in the direction of the generator, and $\bar{q}$ in the direction of the tangent to the contour line of the tangent to the contour line of the cross-section. Setting, according to equations (1.1) and (1.2)

$$u = U_i(z,t)\varphi_i(s) \quad (i = 1,2,\ldots, m), \quad v = V_k(z,t)\psi_k(s) \quad (k = 1,2,\ldots, m) \tag{8.2}$$

Substituting these values in (8.1) and computing the free terms of the equations (2.11) by formulas (2.15) we obtain

$$\begin{aligned} p_j(z,t) &= -\frac{\gamma}{g}\sum \frac{\partial^2}{\partial t^2} U_i(z,t)\int \varphi_i(s)\,\varphi_j(s)\,dF \\ q_h(z,t) &= -\frac{\gamma}{g}\sum \frac{\partial^2}{\partial t^2} V_k(z,t)\int \psi_k(s)\psi_h(s)\,dF \end{aligned} \tag{8.3}$$

Or taking into account formulas (2.13)

$$\begin{aligned} p_j(z,t) &= -\frac{\gamma}{g}\sum a_{ji}\frac{\partial^2}{\partial t^2} U_i(z,t) \qquad (i = 1,2,\ldots, m) \\ q_h(z,t) &= -\frac{\gamma}{g}\sum r_{hk}\frac{\partial^2}{\partial t^2} V_k(z,t) \qquad (k = 1,2,\ldots, n) \end{aligned} \tag{8.4}$$

Equations (2.11) in the case of the dynamic problem will have the form

$$\begin{aligned} &\frac{E}{G}\sum a_{ji}U_i'' - \sum b_{ji}U_i - \frac{\gamma}{g}\sum a_{ji}\ddot{U}_i - \sum c_{jk}V_k' + \frac{1}{G}p_j^{\circ} = 0 \\ &\sum c_{hi}U_i' + \sum r_{hk}V_k'' - \frac{E}{G}\sum s_{hk}V_k - \frac{\gamma}{g}\sum r_{hk}\ddot{V}_k + \frac{1}{G}q_h^{\circ} = 0 \end{aligned} \tag{8.5}$$

where the derivatives with respect to the coordinate $z$ are denoted by primes and those with respect to the time $t$ by dots, $p_j^{o} = p_j^{o}(z,t)$, $q_h^{o} = q_h^{o}(z,t)$ denote the generalized external longitudinal and transverse forces considered as functions of $z$ and $t$ and computed by formulas (2.15). If these forces are absent we have homogeneous equations in partial derivatives referring to the free vibrations of shells. These vibrations, as is seen from equations (8.5), will be harmonic.

Using the method of the separation of variables we obtain a system of homogeneous ordinary linear symmetrical differential equations with a parameter determining the frequency of the natural vibrations of the shell. With these differential equations and the associated homogeneous statical, kinematic, or mixed boundary conditions (depending on the character of the problem), the frequency and the form of the vibrations of the multiply-connected shell are determined as of a spatial, discretely continuous system, the cross-sections of which do not remain plane.

If the boundary conditions are such that for $z = 0$, $z = l$

$$U_i' = 0 \quad (i = 1,2,\ldots, m), \quad V_k = 0 \quad (k = 1,2,\ldots, n) \tag{8.6}$$

the fundamental functions of the homogeneous differential equations will be

$$U_{i\nu}(z,t) = A_i \cos \frac{\nu\pi z}{l} \sin \omega_\nu t, \qquad V_{k\nu}(z,t) = B_k \sin \frac{\nu\pi z}{l} \sin \omega_\nu t \tag{8.7}$$

Substituting (8.7) in equations (8.5) we obtain with $p_j^{o} = q_h^{o} = 0$ for $A_i$, $B_k$ a symmetrical system of homogeneous differential algebraic equations. Equating the determinant of this system to zero we obtain the characteristic equation for the frequency of vibrations $\omega_\nu$. Since the initial equations are symmetrical all the frequencies of vibration will always have real values independently of the shape of contour of the shell cross-section.

The investigations conducted by the author on the vibrations of thin-walled rods and shells show that the fundamental frequency of vibration will always be below that frequency which is obtained on the basis of the existing elementary theory of Rayleigh. In particular, for a shell of the type of a wing of an airplane, this frequency obtained on the basis of the theory here developed may be 3 to 4 times below the frequency obtained from the usual computation.

## 9. General Remarks

The method here developed can be used to obtain the solution of the plane problem of the theory of elasticity for a rectangular region. Thus, for example, for the case of the plane parallel state of the rectangular plate the biharmonic problem leads to a system of symmetrically constructed ordinary differential equations

$$\sum_i U_i'' \iint \varphi_i \varphi_j \, dF - \frac{1-\nu}{2} \sum_i U_i \int \varphi_i' \varphi_j' \, dF$$

$$+ \sum_k V_k' \left( \nu \int \psi_k' \varphi_j \, dF - \frac{1-\nu}{2} \int \psi_k \varphi_j' \, dF \right) + \frac{1-\nu^2}{2E} p_j = 0$$

$$- \sum_i U_i' \left( \nu \int \varphi_i \psi_h' \, dF - \frac{1-\nu}{2} \int \varphi_i' \psi_h \, dF \right) + \frac{1-\nu}{2} \sum_k V_k'' \int \psi_k \psi_h \, dF \quad (9.1)$$

$$- \sum_k V_k \int \psi_k' \psi_h' \, dF + \frac{1-\nu^2}{2E} q_h = 0$$

$$(i,j = 1,2,\ldots, m; \quad k,h = 1,2,\ldots, n)$$

where $\nu$ is the Poisson coefficient, $U_i = U_i(z)$ $(i = 1,2,\ldots m)$, $V_k = V_k(z)$ $(k = 1,2,\ldots n)$ are the required generalized longitudinal and transverse displacements respectively, $\varphi_i = \varphi_i(s)$ $(i = 1,2,\ldots, M)$, $\psi_k = \psi_k(s)$ $(k = 1,2,\ldots, n)$ are given functions depending on the position of the point along the width of the plates and determining according to equation (1.1) the state of the longitudinal and transverse displacements over the section $z = \text{constant}$. The functions $\varphi_i(s)$ and $\psi_k(s)$ on the longitudinal edges of the plates must satisfy only the kinematic conditions. In choosing these functions the rectangular plate can be divided along its width by a number of narrow strips and within the limits of each such strip the assumption of linear distribution of the longitudinal and transverse displacements can be made within the limits of the width of a strip. In the particular case of $m = 1$, $n = 1$ we obtain a system of two differential equations of the second order. These equations for $\varphi_1 = \psi_1 = s$ and for the coordinate $s$ measured from the center point of the width of the plate will refer to the case of the bending of a plate. If the coordinate $s$ is measured from the lower edge of the plate, then for $\varphi_1 = \psi_1 = s$ we shall have equations

referring to a narrow rectangular plate which over its entire lower edge is rigidly fixed against vertical and horizontal displacements, etc.

We may remark, that, in contrast to equations (2.11), equations (9.1) were obtained with account taken of the tensile deformations of the plate over its width. Equations (9.1) in agreement with what was said above represent the generalized equations of the theory of bending of beams and make possible the solution of a number of problems. To such problems for example belong the contact problem on the computation of a beam lying on an elastic base not following the hypothesis of Winkler, the problem of the local stresses in a narrow plate loaded by two concentrated forces applied on one vertical along the longitudinal edges of the plate and directed toward various sides, etc.

For the cylindrical shell of arbitrary contour the differential equations in the variables $u = u(z,s)$, $v = v(z,s)$, in accordance with the assumptions here made for $\delta$ = constant have the form

$$E\frac{\partial^2 u}{\partial z^2} + G\frac{\partial^2 u}{\partial s^2} + G\frac{\partial^2 v}{\partial z \partial s} + P = 0 \tag{9.2}$$

$$G\frac{\partial^2 u}{\partial z \partial s} + G\frac{\partial^2 v}{\partial z^2} + \frac{\delta^2}{12(1 - \nu^2)}\left(\frac{\partial}{\partial s} R \frac{\partial^2}{\partial s^2} + \frac{1}{R}\frac{\partial}{\partial s}\right)\left[\frac{\partial^2}{\partial s^2} R \frac{\partial v}{\partial s} + \frac{\partial}{\partial s}\frac{v}{R}\right] + Q = 0$$

where $P$ and $Q$ are terms depending on the surface load, $R = R(s)$ is the radius of curvature. Equations (9.2) in contrast to the equations of Love have a symmetrical structure. It is not difficult to show that if the cylindrical shell is considered as prismatic and the number of sides is increased to infinity the system (2.11) in the limit passes over into the equations (9.2). Equations (9.2) can be integrated by the method of the separation of variables. Setting (for $P = Q = 0$)

$$u = Z(z)\varphi(s), \qquad v = Z'(z)\psi(s), \qquad Z'' = \lambda^2 Z(z) \tag{9.3}$$

where $\lambda^2$ is an undetermined multiplier, we obtain

$$E\lambda^2\varphi + G\varphi'' + G\psi' = 0$$

$$G\varphi' + G\lambda^2\psi + \frac{\delta^2}{12(1 - \nu^2)}\left(\frac{d}{ds} R \frac{d^2}{ds^2} + \frac{1}{R}\frac{d}{ds}\right)\left[\frac{d^2}{ds^2} R \frac{d}{ds} + \frac{d}{ds}\frac{\psi}{R}\right] = 0 \tag{9.4}$$

The parameter $\lambda$ is found by solving the corresponding homogeneous boundary problem described by the equations (9.4) and the homogeneous boundary conditions on the rectangular edges of the shell. In the case of a shell with rigid contour the problem reduces to the integration of the equations

$$\frac{\partial(\sigma\delta)}{\partial z} + \frac{\partial(\tau\delta)}{\partial s} = 0, \qquad \frac{1}{E}\frac{\partial\sigma}{\partial s} - \frac{1}{G}\frac{\partial\tau}{\partial z} + \xi''x' + \eta''y' + \theta''\omega' = 0 \tag{9.5}$$

for the additional integral conditions

$$\begin{gathered}\int\frac{\partial^2\sigma}{\partial z^2}x\,dF + qx + \left[\frac{dT}{dz}x\right] = 0, \quad \int\frac{\partial^2\sigma}{\partial z^2}y\,dF + q_y + \left[\frac{dT}{dz}y\right] = 0 \\ \int\frac{\partial^2\sigma}{\partial z^2}\omega\,dF + m + \left[\frac{dT}{dz}\omega\right] = 0\end{gathered} \tag{9.6}$$

In equations (9.5) $\xi = \xi(z)$, $\eta = \eta(z)$, and $\theta = \theta(z)$ are respectively the deflections and angle of torsion, $x = x(s)$, $y = y(s)$, $\omega = \omega(s)$ are respectively the cartesian coordinates and the sectorial area. In equations (9.6) the integrals are taken over the entire contour of the section for the variable $dF = \delta ds$, $T = T(z)$ denoting the shearing force.

Translation by S. Reiss
National Advisory Committee
for Aeronautics

REFERENCES

1. Vlasov, V. Z.: Thin-Walled Elastic Rods. Gosstroizdat, 1940.

2. Vlasov, V. Z.: Stroitelnaya Promishlennost. 1931, No. 11, 12.

3. Vlasov, V. Z.: New Method of Computing Prismatic Composite Systems. Gosstroizdat, 1933.

4. Vlasov, V. Z.: The Structural Mechanics of Shells. Gosstroizdat, 1935.

5. Vlasov, V. Z.: Prikladnaya Matematika i Mekhanika, 1944, Vol. VIII, No. 2.

TABLE 1

| $U_1$ | $U_2$ | $\cdots$ | $U_m$ | $V_1$ | $V_2$ | $\cdots$ | $V_n$ | |
|---|---|---|---|---|---|---|---|---|
| $\gamma a_{11}D^2 - b_{11}$ | $\gamma a_{12}D^2 - b_{12}$ | $\cdots$ | $\gamma a_{1m}D^2 - b_{1m}$ | $-c_{11}D$ | $-c_{12}D$ | $\cdots$ | $-c_{1n}D$ | $\frac{1}{G}\,p_1$ |
| $\gamma a_{21}D^2 - b_{21}$ | $\gamma a_{22}D^2 - b_{22}$ | $\cdots$ | $\gamma a_{2m}D^2 - b_{2m}$ | $-c_{21}D$ | $-c_{22}D$ | $\cdots$ | $-c_{2n}D$ | $\frac{1}{G}\,p_1$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $\gamma a_{m1}D^2 - b_{m1}$ | $\gamma a_{m2}D^2 - b_{m2}$ | $\cdots$ | $\gamma a_{mm}D^2 - b_{mm}$ | $-c_{m1}D$ | $-c_{m2}D$ | $\cdots$ | $-c_{mn}D$ | $\frac{1}{G}\,p_m$ |
| $c_{11}D$ | $c_{12}D$ | $\cdots$ | $c_{1m}D$ | $r_{11}D^2 - \gamma s_{11}$ | $r_{12}D^2 - \gamma s_{12}$ | $\cdots$ | $r_{1n}D^2 - \gamma s_{1n}$ | $\frac{1}{G}\,q_1$ |
| $c_{21}D$ | $c_{22}D$ | $\cdots$ | $c_{2m}D$ | $r_{21}D^2 - \gamma s_{21}$ | $r_{22}D^2 - \gamma s_{22}$ | $\cdots$ | $r_{2n}D^2 - \gamma s_{2n}$ | $\frac{1}{G}\,q_1$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $c_{n1}D$ | $c_{n2}D$ | $\cdots$ | $c_{nm}D$ | $r_{n1}D^2 - \gamma s_{n1}$ | $r_{n2}D^2 - \gamma s_{n2}$ | $\cdots$ | $r_{nn}D^2 - \gamma s_{nn}$ | $\frac{1}{G}\,q_n$ |

(2 16)

TABLE 2

| $U_1$ | $U_2$ | ... | $U_m$ | $V_1$ | $V_2$ | ... | $V_n$ | |
|---|---|---|---|---|---|---|---|---|
| $\gamma a_{11}D^2 - b_{11}$ | $-b_{12}$ | ... | $-b_{1m}$ | $-c_{11}D$ | $-c_{12}D$ | ... | $-c_{1n}D$ | $\frac{1}{G}$ |
| $-b_{21}$ | $\gamma a_{12}D^2 - b_{22}$ | ... | $-b_{2m}$ | $-c_{21}D$ | $-c_{22}D$ | ... | $-c_{2n}D$ | $\frac{1}{G}$ |
| ... | ... | ... | ... | ... | ... | ... | ... | ... |
| ... | ... | ... | ... | ... | ... | ... | ... | ... |
| $-b_{m1}$ | $-b_{m2}$ | ... | $\gamma a_{mm}D^2 - b_{mm}$ | $-c_{m1}D$ | $-c_{m2}D$ | ... | $-c_{mn}D$ | $\frac{1}{G}$ |
| $c_{11}D$ | $c_{12}D$ | ... | $c_{1m}D$ | $r_{11}D^2 - \gamma s_{11}$ | $-\gamma s_{12}$ | ... | $-\gamma s_{1n}$ | $\frac{1}{G}$ |
| $c_{21}D$ | $c_{22}D$ | ... | $c_{2m}D$ | $-\gamma s_{21}$ | $r_{22}D^2 - \gamma s_{22}$ | ... | $-\gamma s_{2n}$ | $\frac{1}{G}$ |
| ... | ... | ... | ... | ... | ... | ... | ... | ... |
| ... | ... | ... | ... | ... | ... | ... | ... | ... |
| $c_{n1}D$ | $c_{n2}D$ | ... | $c_{nm}D$ | $-\gamma s_{n1}$ | $-\gamma s_{m2}$ | ... | $r_{mn}D^2 - \gamma s_{nn}$ | $\frac{1}{G}$ |

(2.19)

TABLE 3

| $\zeta = \frac{z}{d}$ | 0 | 1 | 2 | 3 | 4 | 8 | 12 | 10 | $\infty$ |
|---|---|---|---|---|---|---|---|---|---|
| $\frac{E\delta}{P} U_1$ | 0 | 0.516 | 0.639 | 0.626 | 0.570 | 0.334 | 0.189 | 0.108 | 0 |
| $\frac{E\delta}{P} V_1$ | 19.44 | 17.90 | 15.92 | 13.93 | 12.17 | 6.93 | 3.92 | 2.24 | 0 |
| $\frac{F}{P} \sigma$ | 8.64 | 2.48 | .301 | -.426 | -.625 | -.459 | -.261 | -.149 | 0 |
| $\frac{F}{P} \tau$ | -.50 | -.437 | -.381 | -.331 | -.289 | -.164 | -.092 | -.053 | 0 |

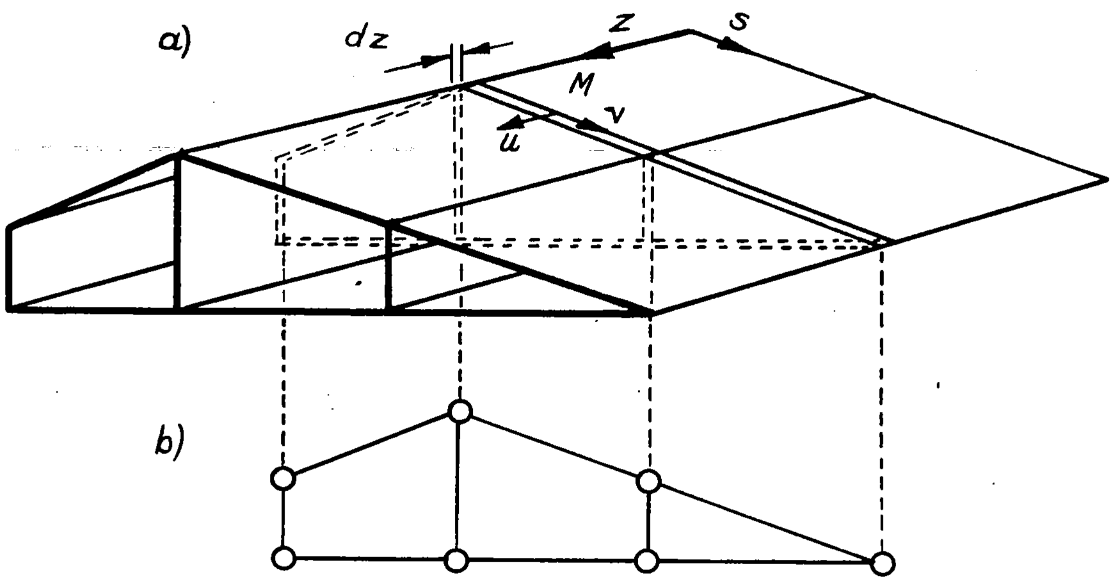
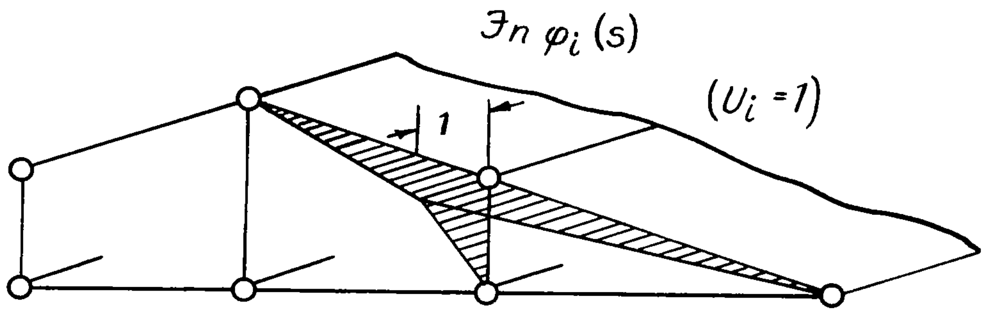



Figure 1.

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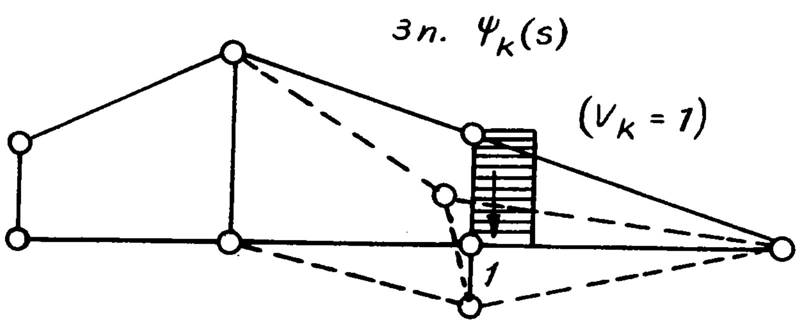


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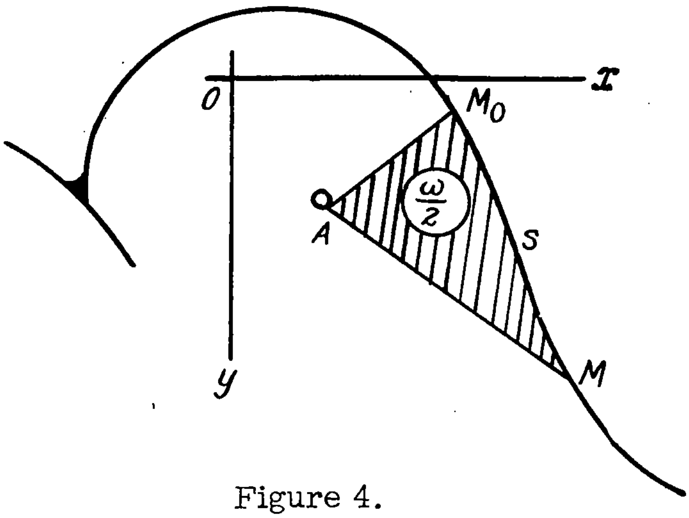


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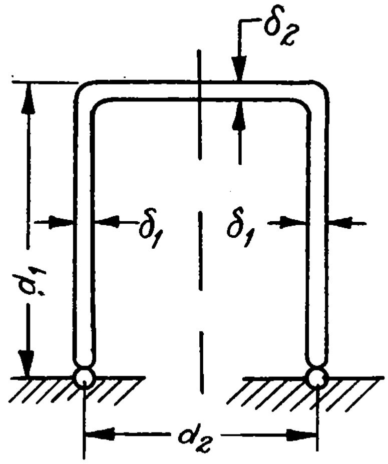


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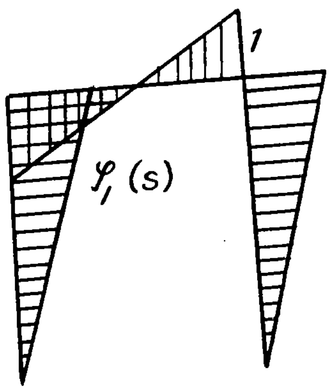


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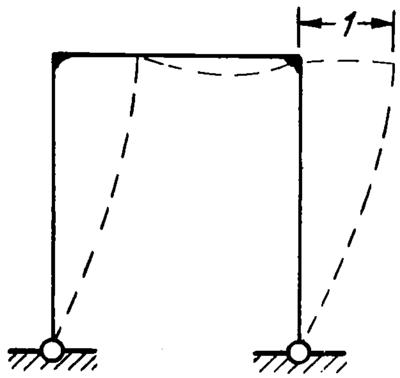


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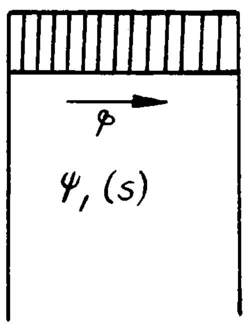


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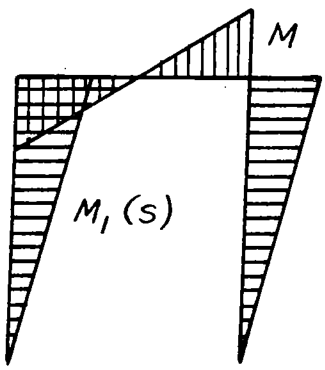


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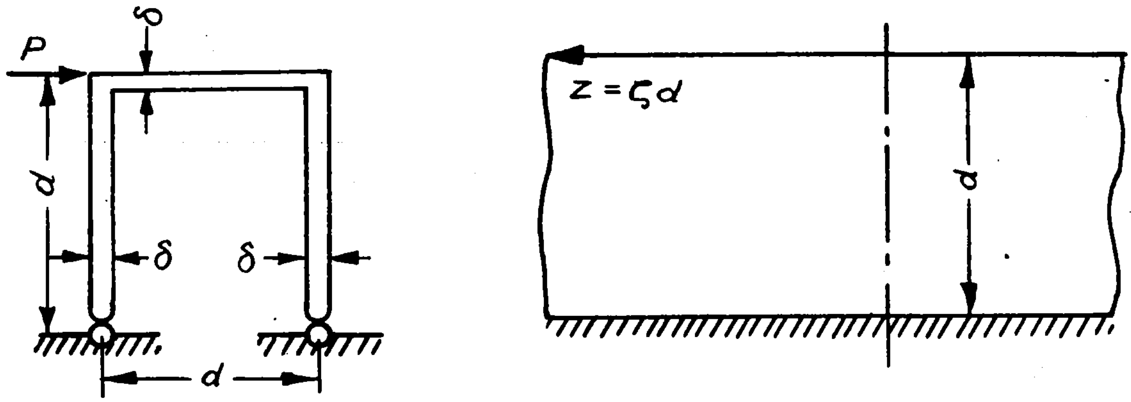


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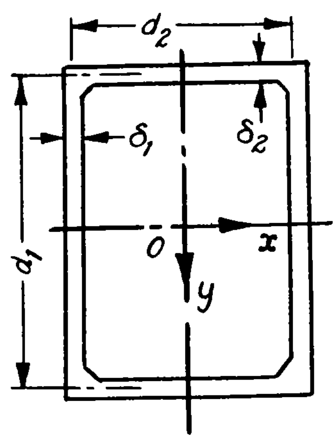


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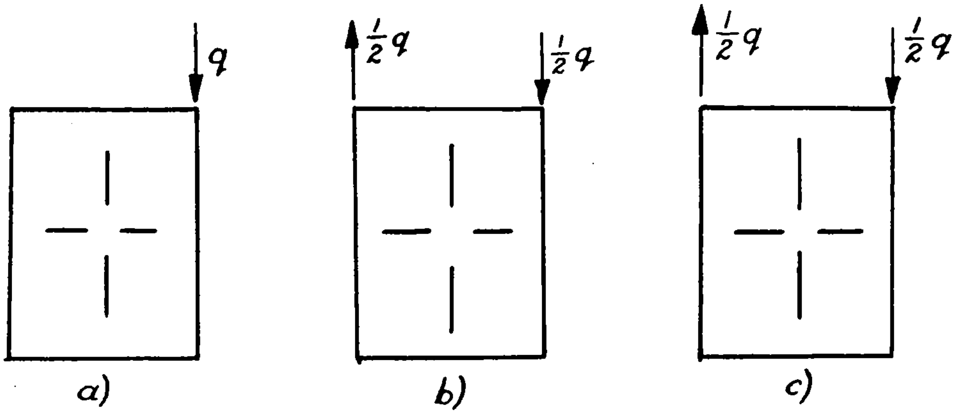


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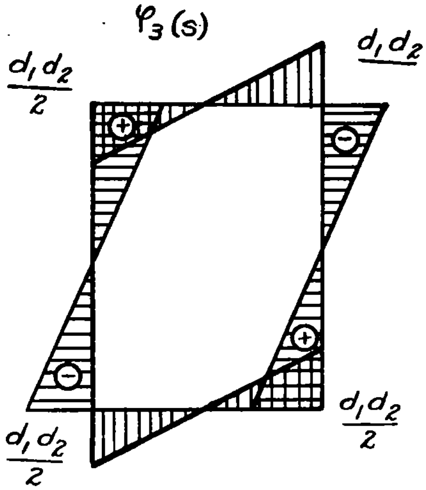


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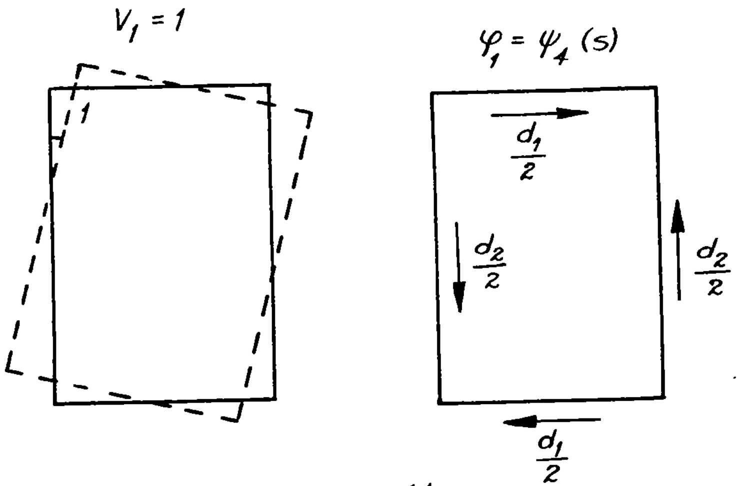


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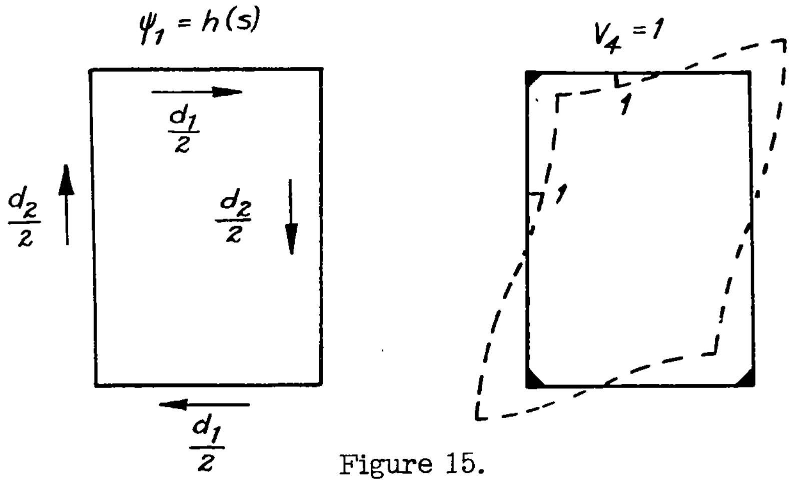


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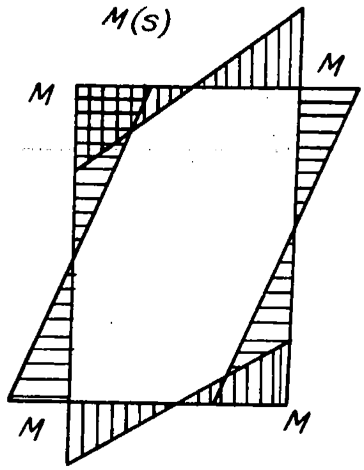


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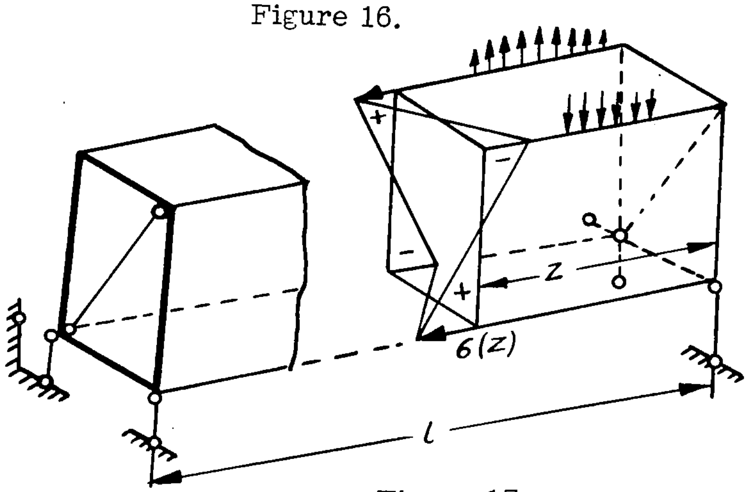


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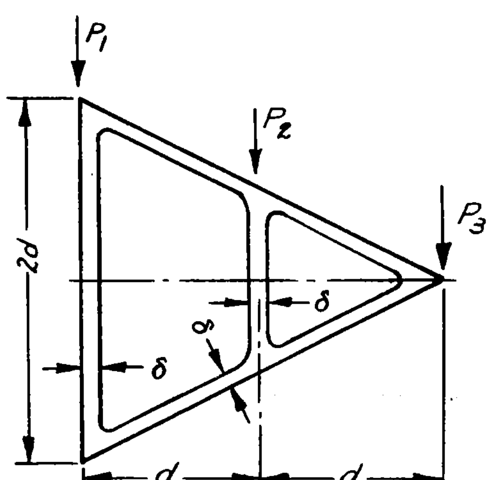


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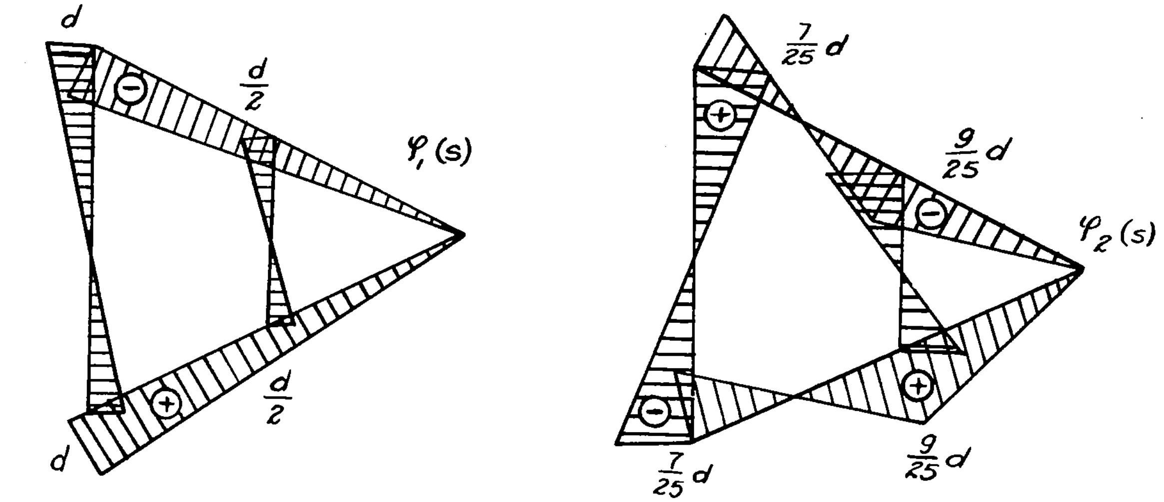


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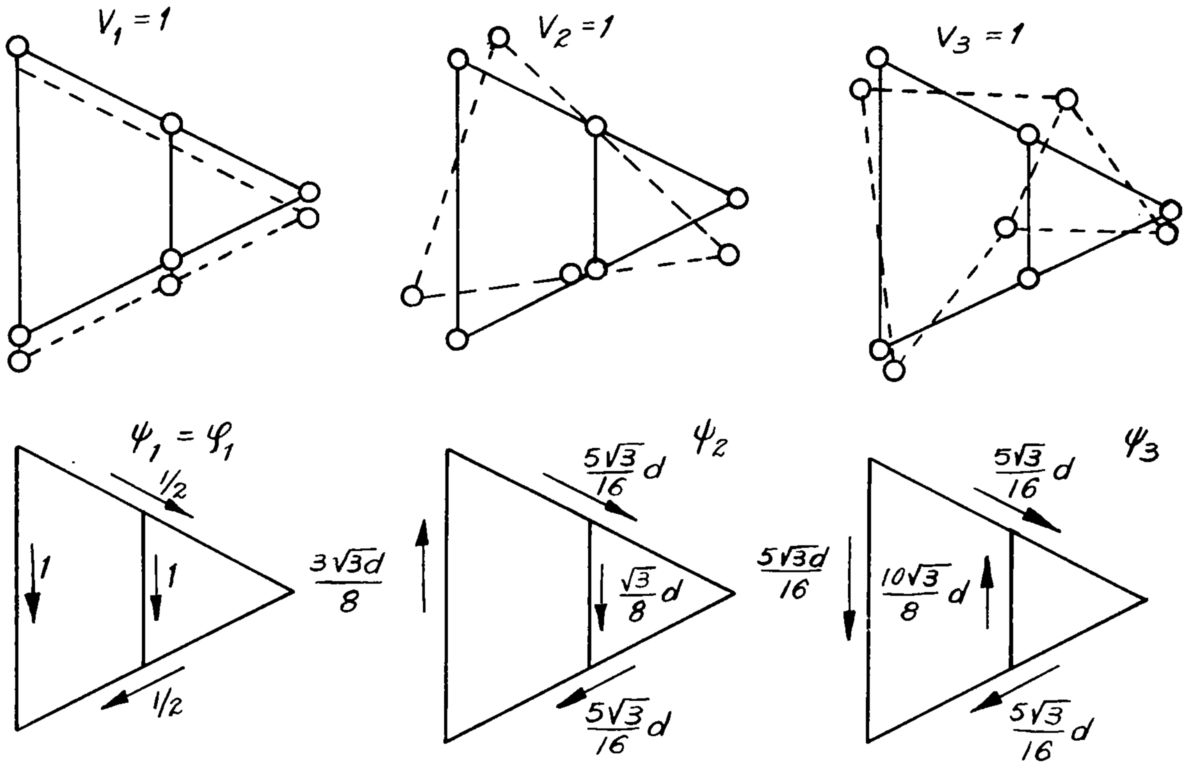


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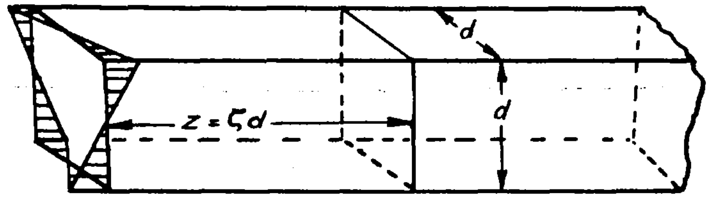


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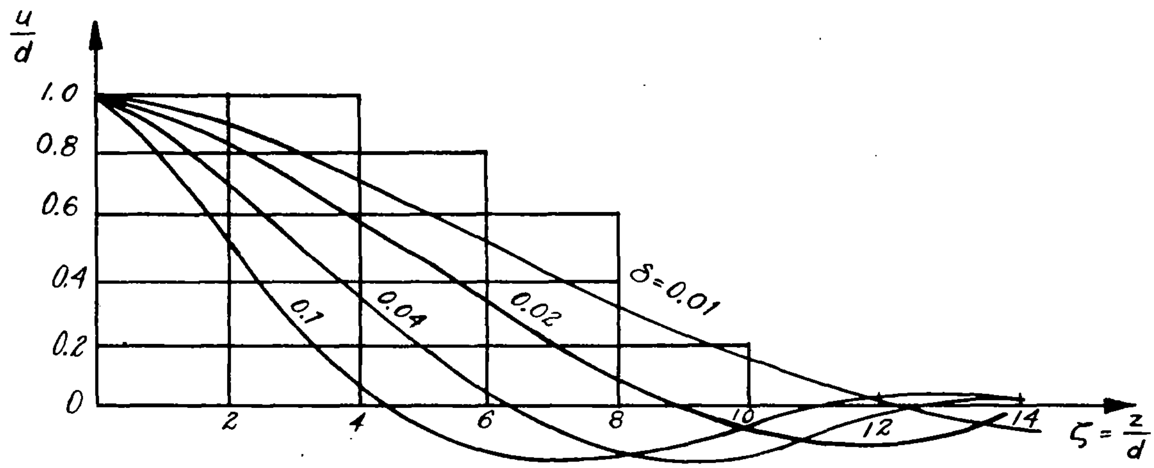


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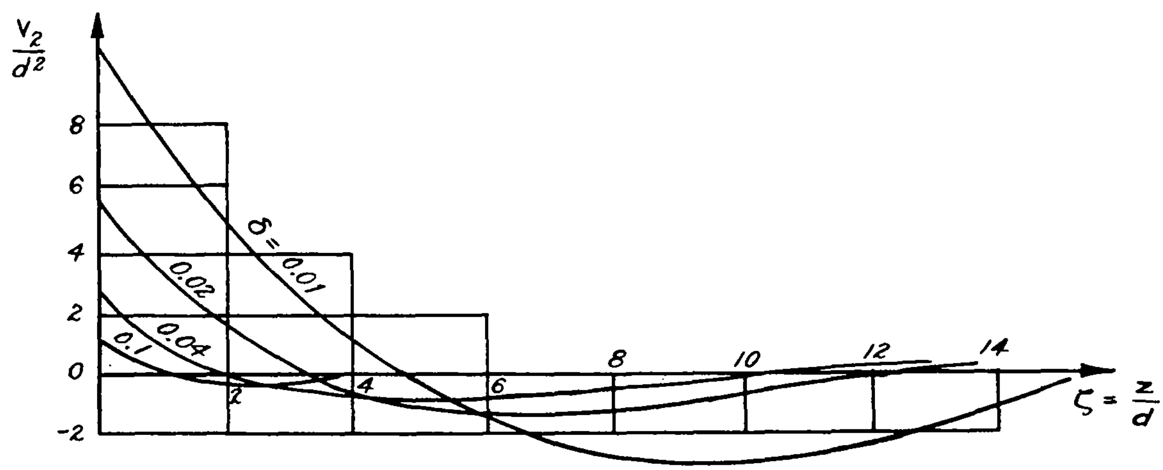


Figure 23.

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